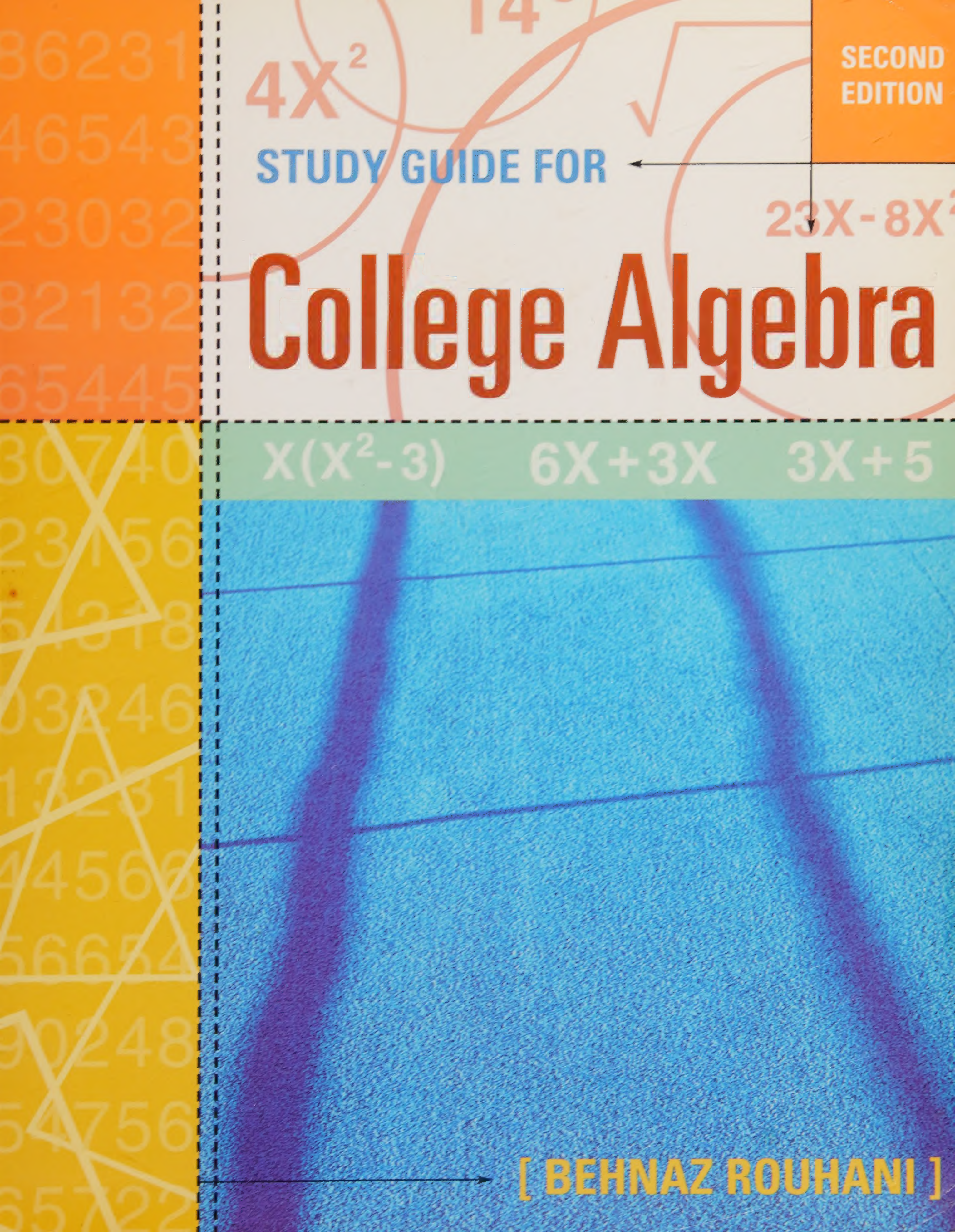

SECOND EDITION

STUDY GUIDE FOR

College Algebra

[BEHNAZ ROUHANI]

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HOUGHTON MIFFLIN COMPANY
Boston New York

Custom Publishing Editor: Kyle Henderson
Custom Publishing Production Manager: Kathleen McCourt

Cover Designer: Ryan Duda
Cover Image: PhotoDisc, Inc.

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Printed in the United States of America.

ISBN: 0-618-19350-2
N00461

1 2 3 4 5 6 7 8 9 – CCI – 04 03 02 01

Houghton Mifflin
Custom Publishing

222 Berkeley Street • Boston, MA 02116

Address all correspondence and order information to the above address.

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APPENDIX **A1**

PREFACE

This study guide is designed to help students develop proficiency in college algebra, and strengthen their understanding of the underlying concepts with ease.

The material has been divided into seven chapters. Each chapter provide in varying detail review of key points and examples. This is followed by sets of solved problems. The general style of the solved problems is to explain all the steps involved, and to provide at least one problem of each type worked in detail. Every chapter is concluded with a practice test to help students evaluate their understanding of the subject.

This study guide helps several groups of students: those who learn by exploring many problems and their solutions; those who need additional help on a particular problem; and those who have studied the material previously and only need to review.

If you have suggestions for improving the text, please feel free to write to me. I do value your view point. Please send comments to: Behnaz Rouhani, Athens Technical College, 800 U.S. Highway 29 North, Athens, Georgia; or by email, to rouhani@aati.edu

ACKNOWLEDGMENT

I would like to thank Dr. Kenneth Jarrett for his unfailing support and understanding during this project. Many thanks also go to several people who assisted me in various capacities: Dr. Frank Churchman for his review of parts of the manuscript and his valuable suggestions; Mrs. Marilyn Edwards for her useful comments, my students for their constant review of the manuscript; Elham Rouhani, my sister, for her constant dedication in trouble shooting the technical aspects of the project; Saba, my daughter, for her love, patience, and sacrifice throughout this project.

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CHAPTER ONE

Fundamental Algebraic Concepts

1.1 The Real Numbers

1.2 Integer Exponents

1.3 Rational Exponents and Radicals

1.4 Polynomials

1.5 Factoring Techniques

1.6 Rational Expressions

1.7 Complex Numbers

1.1 THE REAL NUMBERS

Sets of Numbers

Some important sets of numbers that we study are discussed below:

The numbers we use to count are called the *natural numbers.*

Natural Numbers $= \{1, 2, 3, ...\}$

The *whole numbers* include zero and the natural numbers.

Whole Numbers $= \{0, 1, 2, 3, ...\}$

The *integers* include whole numbers and the negative of natural numbers.

Integers $= \{... -2, -1, 0, 1, 2, ...\}$

The numbers that can be expressed as an integer divided by a nonzero integer are called *rationals.*

Rationals. $= \left\{\frac{a}{b} \mid a \text{ and } b \text{ are integers, } b \neq 0\right\}$

The *irrational numbers* are numbers whose decimal part is nonrepeating and nonterminating. Most square roots have nonrepeating and nonterminating decimals when expressed as decimal numbers.

Irrational Numbers $= \{\pi, \sqrt{2}, \sqrt{7}, \sqrt{5}, ...\}$

The *real numbers* consist of all rational and irrational numbers.

Real Numbers $= \{\mathit{rationals} \text{ and } \mathit{irrationals}\}$

Example 1 **Classify Real numbers**

In the following set which elements are,

a. integers **b.** whole numbers **c.** natural numbers
d. rational numbers **e.** irrational numbers

$\{-3, -0.4, 0, \pi, 8, 83.1, 0.\overline{56}\}$

Solution

a. Integers: $-3, 0, 8$
b. Whole numbers: $0, 8$
c. Natural numbers: 8
d. Rational numbers: $-3, -0.4, 0, 8, 83.1, 0.\overline{56}$
e. Irrational numbers: π

Properties of Real Numbers

If a, b, and c are real numbers,

Closure Property : $a + b$ is a unique real number
ab is a unique real number

Commutative Property : $a + b = b + a$
$ab = ba$

Associative Property : $(a + b) + c = a + (b + c)$
$(ab)\,c = a\,(bc)$

Identity Property : $a + 0 = 0 + a = a$
$a\,(1) = 1\,(a) = a$

Distributive Property : $a\,(b + c) = ab + ac$

Multiplication Property of Zero : $a \cdot 0 = 0$
$0 \cdot a = 0$

Example 2 **Identify Properties of Real Numbers**

Name the property of real numbers illustrated in each of the following statements.

a. $3(ab) = (3a)b$ **b.** $2(x-4) = 2x-8$

c. $x+4 = 4+x$

Solution

a. Associative property of multiplication

b. Distributive property

c. Commutative property of addition

Inverse Property : $a + (-a) = (-a) + a = 0$

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1, \qquad a \neq 0$$

The *additive inverse* of a, that is $-a$, is often called *opposite* of a. Therefore, the additive inverse of 2 is -2.

The *multiplicative inverse* of a, is $\frac{1}{a}$. Therefore, the multiplicative inverse of 3 is $\frac{1}{3}$

Example 3 **Find the Additive inverse and multiplicative inverse**

Find the additive inverse and multiplicative inverse of $-\frac{1}{5}$.

Solution

The additive inverse is $\frac{1}{5}$.

The multiplicative inverse is -5.

Absolute Value

The absolute value of the real number b is the distance between zero and b, and it is denoted by $|b|$.

The formal definition of absolute value is:

$$|b| = \begin{cases} b & if\ b \geq 0 \\ -b & if\ b < 0 \end{cases}$$

The second part of this definition, $|b| = -b$ is misleading. If b is a negative number, then $-b$ is a positive number and $|b|$ will be positive.

DISTANCE BETWEEN TWO POINTS

The distance between two points with coordinates a and b is

$$d(a,b) = |a-b|$$

Example 4 **Find the Distance Between Points**

Find the distance between the points 3 and -4.

Solution

$d(3,-4) = |3-(-4)| = |3+4| = 7$

Operations on Real Numbers

PROPERTIES OF FRACTIONS

If a, b, c, and d are real numbers with $b \neq 0$, $d \neq 0$, then

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$
2. $\frac{ad}{bd} = \frac{a}{b}$
3. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$
4. $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$
5. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
6. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ $, c \neq 0$
7. $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

   $\frac{-a}{-b} = \frac{a}{b}$

DIVISION PROPERTIES OF ZERO

If b is a nonzero number, then

$\frac{0}{b} = 0$

$\frac{b}{0} = undefined$

ORDER OF OPERATIONS

- Perform all operations inside grouping symbols.
- Simplify exponential expressions.
- Perform multiplication and division as they occur from left to right.
- Perform addition and subtraction from left to right.

Example 5 **Evaluate Absolute Value Expression**

Evaluate: $11 - |2 \cdot (-5)| \div 10$

Solution

$$\begin{aligned} 11 - |2 \cdot (-5)| \div 10 &= 11 - |-10| \div 10 \\ &= 11 - 1 = 10 \end{aligned}$$

Example 6 **Evaluate the Expression**

Evaluate: $32 \div (-2) + 3\,(4) + 6$

Solution

$32 \div (-2) + 3\,(4) + 6 = -16 + 12 + 6 = 2$

1.2 INTEGER EXPONENTS

If a is any real number and n is any natural number, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}$$

In the above expression, a is the **base,** n is the **exponent,** and a^n is the **nth power of a.**

For instance,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$
$$-2^4 = -16$$

Note the difference between the two answers. The parentheses in $(-2)^4$indicates that the base is -2, while in the expression -2^4 the base is 2.

PROPERTIES OF INTEGER EXPONENTS

If a and b are nonzero real numbers and m and n are integers, then

Product $\quad a^m \cdot a^n = a^{m+n}$

Quotient $\quad \dfrac{a^m}{a^n} = a^{m-n}$

Power $\quad (a^m)^n = a^{mn}$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Zero Exponent $\quad a^0 = 1$

Negative Exponent $\quad a^{-n} = \dfrac{1}{a^n}$

Example 7 **Simplify Exponential Expressions**

Simplify.

a. $\dfrac{25x^{-5}y^3}{5x^3y^{-2}}$

b. $\dfrac{9(xy)^{-2}}{(-3)^2 x^{-3}y^{-4}}$

Solution

a. Begin by writing each factor without negative exponents.

$$\frac{25x^{-5}y^3}{5x^3y^{-2}} = \frac{5y^3y^2}{x^3x^5} = \frac{5y^5}{x^8}$$

b. Move all factors with negative exponents from the numerator to the denominator, and from the denominator to the numerator.

$$\frac{9(xy)^{-2}}{(-3)^2 x^{-3}y^{-4}} = \frac{9x^3y^4}{9(xy)^2} = \frac{x^3y^4}{x^2y^2} = xy^2$$

Example 8 **Simplify an Exponential Expression**

Simplify. $\left(2x^0y^{-3}\right)^{-2}$

Solution

$$\left(2x^0y^{-3}\right)^{-2} = \left(\frac{1}{2x^0y^{-3}}\right)^2$$

$$= \left(\frac{y^3}{2}\right)^2 = \frac{y^6}{4}$$

Scientific Notation

A number is written in scientific notation if it has the form $\mathbf{a} \times \mathbf{10}^n$, where

$$1 \leq a < 10$$

and n is an integer.

Example 9 **Scientific Notation**

Write each number in scientific notation.

a. $2,035$ **b.** 0.00054

Solution

a. To write $2,035$ in scientific notation, move the decimal point 3 places to the left in order to obtain a number between 1 and 10. The decimal point in $2,035$ is to the right of 5. Since the original number is greater than 10, the exponent is positive.

$2,035 = 2.035 \times 10^3$

b. To write 0.00054 in scientific notation, move the decimal point 4 places to the right in order to obtain a number between 1 and 10. Since the original number is less than 1, the exponent is negative.

$0.00054 = 5.4 \times 10^{-4}$

Example 10 **Decimal Notation**

Write 3.55×10^4 in decimal notation.

Solution

To write the given number in decimal notation, move the decimal to the *right* the same number of places as the exponent. The decimal point is moved to the right since the exponent is positive.

$3.55 \times 10^4 = 35,500$

1.3 RATIONAL EXPONENTS AND RADICALS

Rational Exponents

Definition of $a^{\frac{1}{n}}$

In general $a^{\frac{1}{n}}$ is called the **principal nth root of a.**

If n is an *even* positive integer and $a > 0$, then $a^{\frac{1}{n}}$ is the nonnegative real number such that

$$\left(a^{\frac{1}{n}}\right)^n = a.$$

If n is an *odd* positive integer and a is any real number, then $a^{\frac{1}{n}}$ is the real number such that

$$\left(a^{\frac{1}{n}}\right)^n = a.$$

Example 11 **Evaluate each Expression**

Evaluate.

a. $16^{\frac{1}{2}}$ **b.** $-16^{\frac{1}{2}}$ **c.** $(-32)^{\frac{1}{5}}$ **d.** $(-16)^{\frac{1}{4}}$

Solution

a. $16^{\frac{1}{2}} = 4$ because $\left(4^2\right)^{\frac{1}{2}} = 4$

b. $-16^{\frac{1}{2}} = -\left(4^2\right)^{\frac{1}{2}} = -4$

c. $(-32)^{\frac{1}{5}} = \left(-2^5\right)^{\frac{1}{5}} = -2$

d. $(-16)^{\frac{1}{4}}$ is not a real number. Because there is no real number we can raise to the power of 4 and get -16.

Definition of $a^{\frac{m}{n}}$

If m is an integer, n is a positive integer, and $a^{\frac{1}{n}}$ is a real number, then

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

Example 12 **Evaluate each Expression**

Evaluate $27^{\frac{2}{3}}$.

Solution

$27^{\frac{2}{3}} = \left(27^2\right)^{\frac{1}{3}} = (729)^{\frac{1}{3}} = 9$

$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = (3)^2 = 9$ In general, it is easier to find the root before raising to a power.

PROPERTIES OF RATIONAL EXPONENTS

If a and b are nonzero real numbers, and p and q are rational numbers, then

Product $a^p \cdot a^q = a^{p+q}$

Quotient $\dfrac{a^p}{a^q} = a^{p-q}$

Power $(a^p)^q = a^{pq}$

$$(ab)^q = a^q b^q$$

$$\left(\frac{a}{b}\right)^q = \frac{a^q}{b^q}$$

$$a^{-p} = \frac{1}{a^p}$$

Example 13 **Simplify Exponential Expressions**

Simplify.

a. $\left(\dfrac{27x^4}{3x^6}\right)^{\frac{1}{2}}$ (Assume $x > 0$)

b. $\left(\dfrac{x^{-5}}{32x^{-10}}\right)^{-\frac{2}{5}}$

Solution

a. $\left(\dfrac{27x^4}{3x^6}\right)^{\frac{1}{2}} = \left(\dfrac{9}{x^2}\right)^{\frac{1}{2}} = \dfrac{9^{\frac{1}{2}}}{(x^2)^{\frac{1}{2}}} = \dfrac{3}{x}$

b. Begin by writing every factor inside the parentheses with positive exponent.

$$\left(\frac{x^{-5}}{32x^{-10}}\right)^{-\frac{2}{5}} = \left(\frac{x^{10}}{32x^5}\right)^{-\frac{2}{5}}$$

$$= \left(\frac{x^5}{32}\right)^{-\frac{2}{5}}$$ •Apply $\dfrac{a^p}{a^q} = a^{p-q}$.

$$= \left(\frac{32}{x^5}\right)^{\frac{2}{5}}$$ • Apply $(\frac{a}{b})^{-\frac{m}{n}} = (\frac{b}{a})^{\frac{m}{n}}$.

$$= \left(\frac{2^5}{x^5}\right)^{\frac{2}{5}}$$ • Write $32 = 2^5$.

$$= \frac{4}{x^2}$$ •Apply $(a^p)^q = a^{pq}$.

Radicals

If n is a positive integer and a is a real number, then the principal n^{th} root of a is the real number $\sqrt[n]{a}$.

The symbol $\sqrt[n]{\ }$ is called a radical, a is called the radicand, and n is called the index.

In section 1.2, we defined what is meant by a^n when n is an integer. Now, we will extend the definition to a^n, where n is a rational number.

Radical Notation for $a^{\frac{1}{n}}$
If a is a real number, n is a positive integer, and $a^{\frac{1}{n}}$ is a real number, then
$a^{\frac{1}{n}} = \sqrt[n]{a}$.
For example, $(3xy^2)^{\frac{1}{3}} = \sqrt[3]{3xy^2}$

Radical Notation for $a^{\frac{m}{n}}$
If a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.
For instance, $(2x)^{\frac{2}{3}} = \sqrt[3]{(2x)^2} = (\sqrt[3]{2x})^2$

Definition of $\sqrt[n]{a^n}$
If $n \geq 2$ is a positive integer and a is a real number, then
$\sqrt[n]{a^n} = a$ if n is odd
$\sqrt[n]{a^n} = |a|$ if n is even (if $a > 0$, $\sqrt[n]{a^n} = a$)

Example 14 **Find the nth Root of each Expression**

Find each root.

a. $\sqrt[4]{16}$ **b.** $\sqrt{-81}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt[7]{-\frac{1}{128}}$ **e.** $\left(\sqrt[3]{-8}\right)$

Solution

a. $\sqrt[4]{16} = \sqrt[4]{(2)^4} = 2$

b. $\sqrt{-81}$ is not real.

c. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$

d. $\sqrt[7]{-\frac{1}{128}} = \sqrt[7]{\left(-\frac{1}{2}\right)^7} = -\frac{1}{2}$

e. $\left(\sqrt[3]{-8}\right) = \sqrt[3]{(-2)^3} = -2$

PROPERTIES OF RADICALS

If m and n are positive integers ≥ 2 and a and b are nonnegative real numbers, then

Product $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Quotient $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad (b \neq 0)$

Index $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

$$\left(\sqrt[n]{a}\right)^n = a$$

$$\sqrt[n]{a^n} = a$$

Example 15 **Simplify Radical Expressions**

Simplify.

a. $\sqrt[4]{48}$

b. $\sqrt[3]{-8x^3y^6z^5}$

c. $\sqrt[4]{16x^{28}y^4}$

Solution

a. $\sqrt[4]{48} = \sqrt[4]{2^4\,(3)} = 2\sqrt[4]{3}$

b. Express each factor as a power of the index, then simplify.

$$\sqrt[3]{-8x^3y^6z^5} = \sqrt[3]{(-2xy^2z)^3\, z^2} = -2xy^2z\sqrt[3]{z^2}$$

c. Express each factor as a perfect 4^{th} power, then simplify.

$$\sqrt[4]{16x^{28}y^4} = \sqrt[4]{2^4\,(x^7)^4\, y^4}$$

$$= 2x^7y$$ • Simplify using $\sqrt[n]{a^n} = a$.

OPERATIONS ON RADICALS

Addition and Subtraction of Radicals: Radicals with the same index and radicand can be combined.

Example 16 **Combine Radical Expressions**

Perform the indicated operations.

a. $3\sqrt{49x^3} - 2x\sqrt{25x}$

b. $2x\sqrt[3]{24x^3} - \sqrt[3]{81x^6}$

Solution

a. $3\sqrt{49x^3} - 2x\sqrt{25x} = 3\sqrt{(7x)^2\,x} - 2x\sqrt{5^2x}$

$= 3\,(7x)\,\sqrt{x} - 2x\,(5)\,\sqrt{x}$

$= 21x\sqrt{x} - 10x\sqrt{x} = 11x\sqrt{x}$

b. $2x\sqrt[3]{24x^3} - \sqrt[3]{81x^6} = 2x\sqrt[3]{3\cdot 2^3\cdot x^3} - \sqrt[3]{3^3\cdot 3\cdot (x^2)^3}$

$= 2x\,(2)\,(x)\,\sqrt[3]{3} - (3)\,(x^2)\,\sqrt[3]{3}$

$= 4x^2\sqrt[3]{3} - 3x^2\sqrt[3]{3}$

$= x^2\sqrt[3]{3}$

Multiplication of Radicals: Radicals with the same indices can be multiplied.

Example 17 **Multiply Radical Expressions**

Perform the indicated operation.

a. $\sqrt[3]{-9xy}\cdot\sqrt[3]{3x^2y^2}$

b. $\left(\sqrt{5}+2\right)\left(\sqrt{5}-4\right)$

Solution

a. $\sqrt[3]{-9xy}\cdot\sqrt[3]{3x^2y^2} = \sqrt[3]{-27x^3y^3} = -3xy$

b. $\left(\sqrt{5}+2\right)\left(\sqrt{5}-4\right) = \sqrt{25} - 4\sqrt{5} + 2\sqrt{5} - 8$

$= 5 - 2\sqrt{5} - 8$

$= -3 - 2\sqrt{5}$

Division of Radicals: Division of radicals is performed by rationalizing the denominator. The process of removing radicals from the denominator is called *rationalizing the denominator.*

Example 18 **Rationalize the Denominator**

Rationalize the denominator.

a. $\dfrac{2x}{\sqrt{3x}}$

b. $\sqrt[3]{\dfrac{30x}{6x^2}}$

c. $\dfrac{x}{\sqrt[4]{9x}}$

Solution

a. $\dfrac{2x}{\sqrt{3x}} = \dfrac{2x}{\sqrt{3x}}\cdot\dfrac{\sqrt{3\mathbf{x}}}{\sqrt{3\mathbf{x}}} = \dfrac{2x\sqrt{3x}}{3x} = \dfrac{2\sqrt{3x}}{3}$

b. $\sqrt[3]{\dfrac{30x}{6x^2}} = \sqrt[3]{\dfrac{5}{x}} = \dfrac{\sqrt[3]{5}}{\sqrt[3]{x}} = \dfrac{\sqrt[3]{5}}{\sqrt[3]{x}}\cdot\dfrac{\sqrt[3]{\mathbf{x^2}}}{\sqrt[3]{\mathbf{x^2}}} = \dfrac{\sqrt[3]{5x^2}}{x}$

c. $\dfrac{x}{\sqrt[4]{9x}} = \dfrac{x}{\sqrt[4]{3^2\cdot x}}\cdot\dfrac{\sqrt[4]{\mathbf{3^2\cdot x^3}}}{\sqrt[4]{\mathbf{3^2\cdot x^3}}}$

We multiplied the numerator and the denominator by $\sqrt[4]{3^2x^3}$ because $\sqrt[4]{3^2x} \cdot \sqrt[4]{3^2x^3} = \sqrt[4]{3^4x^4} = 3x$.

$$= \frac{x\sqrt[4]{9x^3}}{\sqrt[4]{3^4 \cdot x^4}}$$

$$= \frac{x\sqrt[4]{9x^3}}{3x}$$

$$= \frac{\sqrt[4]{9x^3}}{3}$$

Example 19 **Rationalize the Denominator**

Rationalize the denominator.

a. $\dfrac{2}{\sqrt{3}-\sqrt{5}}$

b. $\dfrac{2-3\sqrt{2}}{3-\sqrt{5}}$

Solution

a. $\dfrac{2}{\sqrt{3}-\sqrt{5}} = \dfrac{2}{\sqrt{3}-\sqrt{5}} \cdot \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}}$

$= \dfrac{2\sqrt{3}+2\sqrt{5}}{3-5}$

$= \dfrac{2\sqrt{3}+2\sqrt{5}}{-2}$ • Divide each term in the numerator and denominator by 2.

$= -\sqrt{3}-\sqrt{5}$

b. $\dfrac{2-3\sqrt{2}}{3-\sqrt{5}} = \dfrac{2-3\sqrt{2}}{3-\sqrt{5}} \cdot \dfrac{3+\sqrt{5}}{3+\sqrt{5}}$

$= \dfrac{6+2\sqrt{5}-9\sqrt{2}-3\sqrt{10}}{9-5}$

$= \dfrac{6+2\sqrt{5}-9\sqrt{2}-3\sqrt{10}}{4}$

1.4 POLYNOMIALS

A **polynomial** is an algebraic expression in which no variable appears in any denominator or has rational number exponents.

A **term** is a constant, a variable, or a product of a constant and a variable.

The **degree of a polynomial** is the degree of the term with the highest degree.

The **degree of a term** is the sum of the exponents of the variable.

A polynomial containing three terms is called a **trinomial;** the one with two terms is a **binomial;** and a single term polynomial is called **monomial.**

The list below gives the degree of each polynomial and identifies each polynomial.

Polynomial	**Degree**	**Type**
$4x^4 - 6x^3 + x$	4	Trinomial
$3x^7 - 9x^5$	7	Binomial
$-6x^5y^7$	12	Monomial
$3x^3y^2 - x^5y^2 + y^8 - 7x^{12}y$	13	None

Operations with Polynomials

ADDITION AND SUBTRACTION

To add or subtract polynomials, combine like terms, and write the result in simplest form.

Example 20 **Combine Polynomials**

Simplify. $(x^2 - 2x + 5) - (3x^2 - 6x - 2)$

Solution

$$\begin{aligned}&(x^2 - 2x + 5) - (3x^2 - 6x - 2)\\&= x^2 - 2x + 5 - 3x^2 + 6x + 2\\&= -2x^2 + 4x + 7\end{aligned}$$

MULTIPLICATION

To find the product of two polynomials, multiply each term of the first polynomial by each term of the second polynomial, then combine like terms, and write the result in simplest form.

Example 21 **Multiply Polynomials**

Perform the indicated operation. $(2x - 3)\left(2x^2 - 4x + 5\right)$

Solution

$$\begin{aligned}&(2x - 3)\left(2x^2 - 4x + 5\right)\\&= 2x\left(2x^2 - 4x + 5\right) - 3\left(2x^2 - 4x + 5\right)\\&= 4x^3 - 8x^2 + 10x - 6x^2 + 12x - 15\\&= 4x^3 - 14x^2 + 22x - 15\end{aligned}$$

1.5 FACTORING TECHNIQUES

Factoring Formulas

The special products listed below, can be regarded as patterns for factoring.

DIFFERENCE OF TWO SQUARES: $a^2 - b^2 = (a-b)(a+b)$
PERFECT SQUARE TRINOMIALS: $a^2 + 2ab + b^2 = (a+b)^2$
$a^2 - 2ab + b^2 = (a-b)^2$
SUM OF TWO CUBES: $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$
DIFFERENCE OF TWO CUBES: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Factoring by Grouping

If a polynomial has more than three terms, factor using the following steps:

Step 1 Group terms that contain a common factor.
Step 2 Factor out the common factor of each group.
Step 3 Factor out the common factor in the expression resulted in step 2.

Example 22 **Factor by Grouping**

Factor: $8 + 16x - x^3 - 2x^4$

Solution

$8 + 16x - x^3 - x^4$
$= 8(1+2x) - x^3(1+2x)$
$= (1+2x)\left(8 - x^3\right)$
$= (1+2x)(2-x)\left(4 + 2x + x^2\right)$

Factoring Trinomials

Points to Remember to Factor $x^2 + bx + c$

1. Write the trinomial in the factored form $(x+m)(x+n)$.
2. Find the pair of numbers m and n in such a way that,
   $m \cdot n = c$
   and
   $m + n = b$
3. Decide on signs of m and n.
   If b and c are positive, then m and n are both positive.
   If b is negative and c is positive, then m and n are both negative.
   If b and c are negative, then of m and n the one with larger magnitude has negative sign.
   If b is positive and c is negative, then of m and n the one with larger magnitude has positive sign.
4. If there are no such numbers, the polynomial is prime.
5. Check the factoring by multiplying the binomial factors to see if the product is the given trinomial.

Example 23 **Factor a Trinomial of the Form $x^2 + bx + c$**

Factor: $x^2 + 2x - 8$

Solution

Determine all factors of -8, then find their sums.

Factors of -8	Sum of the Factors
$-4, +2$	-2
$\mathbf{-2, +4}$	$\mathbf{+2}$
$-1, +8$	$+7$
$+1, -8$	-7

In the above list the two numbers whose sum is 2 and whose product is -8, are -2 and 4. Therefore,

$$x^2 + 2x - 8 = (x + 4)(x - 2)$$

Points to Remember to Factor $ax^2 + bx + c$

1. List all possible pairs of factors of a.

2. List all possible pairs of factors of c.

3. Try various combinations of these factors until the middle term b is found.

4. If such combination does not exist, the polynomial is prime.

Example 24 **Factor a Trinomial of the Form $ax^2 + bx + c$**

Factor: $6x^2 - 13x + 5$

Solution

We will list all possible factors of 6 and 5 which are the first and last terms respectively.

Factors of 6	Factors of 5
$3, 2$	$-1, -5$
$1, 6$	$-1, -5$

We use these factors to write trial factors.

Trial Factors	Middle Term
$(3x - 1)(2x - 5)$	$-17x$
$\mathbf{(2x - 1)(3x - 5)}$	$\mathbf{-13x}$
$(x - 1)(6x - 5)$	$-11x$
$(x - 5)(6x - 1)$	$-31x$

Therefor, $6x^2 - 13x + 5 = (2x - 1)(3x - 5)$

General Factoring Procedures

Points to Remember

1. Factor out the greatest common factor (GCF) from each term.

2. If the polynomial is a binomial in the form of the difference of two perfect squares or sum/difference of two cubes, then use factoring formulas.

3. If the polynomial is a trinomial, try to factor as a perfect square trinomial using factoring formulas, otherwise use the trial - and - error method.

4. If the polynomial has more than 3 terms, factor by grouping.

5. If the polynomial has degree more than 3, factor using auxiliary variables. This topic will be discussed in chapter 2.

6. Make sure that each factor is factored completely.

Example 25 **Factor a Trinomial**

Factor: $8x^2 - 24x + 18$

Solution

First, factor out the common factor 2 to obtain

$8x^2 - 24x + 18 = 2\left(4x^2 - 12x + 9\right)$

The trinomial $4x^2 - 12x + 9$ can be factored as the product of two binomials. The first term is a square, $(2x)^2$, as is the last term, $(3)^2$. Since $2\,(2x)\,(3) = 12x$, we can factor as follows:

$4x^2 - 12x + 9 = (2x - 3)^2$

Therefore,

$8x^2 - 24x + 18 = 2\,(2x - 3)^2$

1.6 RATIONAL EXPRESSIONS

A **rational expression** is an expression of the form $\frac{P}{Q}$, where P and Q are polynomials and Q is not equal to zero.

The **domain of a rational expression** excludes those values of x that make the denominator of the rational expression zero.

Example 26 **Find the Domain of an Expression**

Find the domain: $\frac{2x-5}{x-3}$

Solution

To find the domain set the denominator equal to zero and solve for x.

$Domain = \{x \mid x \neq 3\}$

PROPERTIES OF RATIONAL EXPRESSIONS

If P, Q, and R are polynomials, and $Q \neq 0, S \neq 0, R \neq 0$, then

$\frac{P}{Q} = \frac{R}{S}$ if and only if $PS = QR$

$$\frac{PR}{QR} = \frac{P}{Q}$$

$$-\frac{P}{Q} = \frac{-P}{Q} = \frac{P}{-Q}$$

SIMPLIFYING RATIONAL EXPRESSIONS

Points to Remember for Simplifying Rational Expressions

1. Factor the numerator and denominator completely.

2. Divide the numerator and denominator by any common factor to reduce to lowest terms.

3. The resulting expression is the answer.

Example 27 **Simplify a Rational Expression**

Simplify: $\frac{x^3+8}{x^3-2x^2+4x}$

Solution

$$\frac{x^3+8}{x^3-2x^2+4x}$$

$$= \frac{(x+2)(x^2-2x+4)}{x(x^2-2x+4)}$$

$$= \frac{x+2}{x} \qquad x \neq 0$$

Caution: Avoid these common errors in simplifying rational expressions.

$\frac{x+2}{x} = 2$ **Incorrect!**

$\frac{x^2+3}{x} = x+3$ **Incorrect!**

$\frac{2}{x+2} = \frac{1}{x}$ **Incorrect!**

$\frac{x^2y - 2x}{y} = x^2 - 2x$ **Incorrect!**

$\frac{x^3+1}{x+1} = x^2$ **Incorrect!**

You can only divide out, or cancel common factors that are parts of products.

OPERATIONS WITH RATIONAL EXPRESSIONS

ADDITION AND SUBTACTION

If P, Q, and R are polynomials, and $Q \neq 0$, then

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$$
$$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$$

Addition or subtraction of rational expressions with **different denominators** is accomplished by first finding the least common denominator, and then writing the sum or difference of the numerators over the common denominator.

Determining The Least Common Denominator

1. Factor each denominator completely.

2. The least common denominator is the product of all different factors from each denominator. If a factor occurs in more than one denominator, the highest power of that factor is used.

Example 28 **Combine Rational Expressions**

Perform the indicated operations.

a. $\frac{x-3}{x^2+3x} - \frac{x}{x^2+6x+9}$

b. $\frac{x^2}{(x-2)^3} - \frac{2}{4-4x+x^2} - \frac{1}{x-2}$

Solution

a. Since $x^2+3x = x(x+3)$ and $x^2+6x+9 = (x+3)^2$, the LCD is $x(x+3)^2$. Therefore,

$$\frac{x-3}{x^2+3x} - \frac{x}{x^2+6x+9}$$
$$= \frac{\mathbf{x+3}}{\mathbf{x+3}} \cdot \frac{x-3}{x(x+3)} - \frac{\mathbf{x}}{\mathbf{x}} \cdot \frac{x}{(x+3)^2}$$
$$= \frac{(x+3)(x-3) - x^2}{x(x+3)^2}$$
$$= \frac{x^2 - 9 - x^2}{x(x+3)^2}$$

$$= \frac{-9}{x(x+3)^2}$$

b. Since $4 - 4x + x^2 = (x-2)^2$, the LCD is $(x-2)^3$.

$$\begin{aligned}
&\frac{x^2}{(x-2)^3} - \frac{2}{4-4x+x^2} - \frac{1}{x-2} = \frac{x^2}{(x-2)^3} - \frac{2}{(x-2)^2} - \frac{1}{x-2}\\
&= \frac{x^2}{(x-2)^3} - \frac{\mathbf{x-2}}{\mathbf{x-2}} \cdot \frac{2}{(x-2)^2} - \frac{(\mathbf{x-2})^2}{(\mathbf{x-2})^2} \cdot \frac{1}{x-2}\\
&= \frac{x^2 - 2(x-2) - (x-2)^2}{(x-2)^3}\\
&= \frac{x^2 - 2x + 4 - (x^2 - 4x + 4)}{(x-2)^3}\\
&= \frac{x^2 - 2x + 4 - x^2 + 4x - 4}{(x-2)^3}\\
&= \frac{2x}{(x-2)^3}
\end{aligned}$$

MULTIPLICATION AND DIVISION

If P, Q, and R are polynomials and $Q \neq 0, S \neq 0$, then

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$$

Example 29 **Multiply and Divide Rational Expresions**

Perform the indicated operations.

a. $\dfrac{9x^2 - y^2}{3x^2 - 2xy - y^2} \cdot \dfrac{x^2 - y^2}{9x - 3y}$

b. $\dfrac{x^3 + 64}{x^2 - 16} \div \dfrac{2x^2 - 8x + 32}{2x - 8}$

Solution

a. Begin by factoring the numerators and denominators completely, then simplify.

$$\frac{9x^2 - y^2}{3x^2 - 2xy - y^2} \cdot \frac{x^2 - y^2}{9x - 3y} = \frac{(3x-y)(3x+y)}{(x-y)(3x+y)} \cdot \frac{(x-y)(x+y)}{3(3x-y)}$$

$$= \frac{x+y}{3}$$

b. First, invert the divisor. Next, factor each rational expression completely. Finally, divide out common factors.

$$\frac{x^3 + 64}{x^2 - 16} \div \frac{2x^2 - 8x + 32}{2x - 8} = \frac{x^3 + 64}{x^2 - 16} \cdot \frac{2x - 8}{2x^2 - 8x + 32}$$

$$= \frac{(x+4)\left(x^2-4x+16\right)}{(x-4)(x+4)} \cdot \frac{2(x-4)}{2\left(x^2-4x+16\right)}$$

$$= 1$$

SIMPLIFYING COMPLEX FRACTIONS

Points to Remember for Simplifying Complex Fractions

1. Simplify the numerator and denominator separately to obtain one fraction in numerator and one fraction in denominator.

2. Multiply outer expressions and place the result in numenator, multiply the inner expressions and put it in the denominator.

3. Write the resulting expression in lowest terms.

Example 30 **Simplify Complex Fractions**

Simplify: $\dfrac{\dfrac{1}{x}+\dfrac{3}{x-3}}{\dfrac{4x}{x-4}-\dfrac{3}{x-4}}$

Solution

$$\frac{\dfrac{1}{x}+\dfrac{3}{x-3}}{\dfrac{4x}{x-4}-\dfrac{3}{x-4}} = \frac{\dfrac{\mathrm{x}-3}{\mathrm{x}-3}\cdot\dfrac{1}{x}+\dfrac{\mathrm{x}}{\mathrm{x}}\cdot\dfrac{3}{x-3}}{\dfrac{4x-3}{x-4}}$$

$$= \frac{\dfrac{x-3}{x(x-3)}+\dfrac{3x}{x(x-3)}}{\dfrac{4x-3}{x-4}}$$

$$= \frac{\dfrac{x-3+3x}{x(x-3)}}{\dfrac{4x-3}{x-4}}$$

$$= \frac{\dfrac{4x-3}{x(x-3)}}{\dfrac{4x-3}{x-4}}$$

$$= \frac{(4x-3)(x-4)}{x(x-3)(4x-3)} = \frac{x-4}{x(x-3)}$$

1.7 COMPLEX NUMBERS

Definition of i

The number i is called the imaginary unit and is defined as

$$i = \sqrt{-1} \qquad \text{where} \qquad i^2 = -1$$

Definition of $\sqrt{-a}$

If a is a real number, then

$$\sqrt{-a} = i\sqrt{a}$$

Note: The expression $\sqrt{a}i$ is often written as $i\sqrt{a}$ so that it is not mistaken for $\sqrt{ai}$.

Definition: If a and b are real numbers and i is the imaginary unit, then $a + bi$ is called a **complex number**. The real number a is called the **real part** of the complex number, and the real number b is called the **imaginary part** of the complex number. A complex number written in the form $a + bi$ is in *standard form.*

Example 31 **Write Complex Number in Standard Form**

Write each complex number in standard form.

a. $2 - \sqrt{-3}$

b. $\sqrt{-27} + 2$

Solution

Use the definition $\sqrt{-a} = i\sqrt{a}$

a. $2 - \sqrt{-3} = 2 - i\sqrt{3}$

b. $\sqrt{-27} + 2 = i\sqrt{27} + 2 = 2 + 3i\sqrt{3}$

OPERATIONS ON COMPLEX NUMBERS

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi) + (c + di) = (a + c) + (b + d)\,i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)\,i$$

Example 32 **Combine Complex Numbers**

Perform the indicated operation, and write answer in standard form.

$(1 - 7i) - (6 - 3i)$

Solution

$(1 - 7i) - (6 - 3i) = 1 - 7i - 6 + 3i = -5 - 4i$

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi)\,(c + di) = (ac - bd) + (ad + bc)\,i$$

Example 33 **Multiply Complex Numbers**

Simplify and write answer in standard form.

a. $(2 + 3i)\,(5 - 4i)$

b. $-2i\,(7 - 2i)$

Solution

a. $(2+3i)(5-4i) = 10-8i+15i-12i^2$
$= 10+7i-12(-1)$
$= 10+7i+12$
$= 22+7i$

b. $-2i(7-2i) = -14i+4i^2 = -14i-4 = -4-14i$

CAUTION! Pay attention to the following correct and incorrect uses of radicals.

Correct : $\sqrt{-a}\cdot\sqrt{-a} = i\sqrt{a}\cdot i\sqrt{a}$
$= i^2\sqrt{a^2}$
$= -a$

Incorrect : $\sqrt{-a}\cdot\sqrt{-a} = \sqrt{a^2}$
$= a$

Example 34 **Operations Involving Square Roots of Negative Numbers**

Perform the indicated operations.

a. $\sqrt{-4}\sqrt{-25}$

b. $\left(2+\sqrt{-3}\right)\left(2-\sqrt{-3}\right)$

c. $\left(2+\sqrt{-7}\right)^2$

Solution:

a. $\sqrt{-4}\sqrt{-25} = (2i)(5i) = 10i^2 = 10(-1) = -10$

b. $\left(2+\sqrt{-3}\right)\left(2-\sqrt{-3}\right) = \left(2+i\sqrt{3}\right)\left(2-i\sqrt{3}\right) = 4-3i^2 = 4-3(-1) = 7$

c. $\left(2+\sqrt{-7}\right)^2 = \left(2+i\sqrt{7}\right)^2$
$= (2)^2 + 2(2)\left(i\sqrt{7}\right) + \left(i\sqrt{7}\right)^2$
$= 4+4i\sqrt{7}+7i^2$
$= 4+4i\sqrt{7}-7$
$= -3+4i\sqrt{7}$

The complex numbers $a+bi$ and $a-bi$ are called **complex conjugates.**
If $a+bi$ and $c+di$ are complex numbers, then

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{c^2+d^2}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

Example 35 **Divide Complex Numbers**

Perform the indicated operation, and write answer in standard form.

a. $\dfrac{1}{3-4i}$

b. $\dfrac{2-3i}{4+2i}$

Solution

a. Multiply the numerator and denominator by the conjugate of the denominator which is $3+4i$.

$$\frac{1}{3-4i}=\frac{1}{3-4i}\cdot\frac{\mathbf{3+4i}}{\mathbf{3+4i}}$$

$$=\frac{3+4i}{9-16i^2}$$

$$=\frac{3+4i}{9+16}$$ • Use $i^2=-1$.

$$=\frac{3}{25}+\frac{4}{25}i$$ • Write answer in standard form.

b. Multiply the numerator and denominator by the conjugate of the denominator which is $4-2i$.

$$\frac{2-3i}{4+2i}=\frac{2-3i}{4+2i}\cdot\frac{\mathbf{4-2i}}{\mathbf{4-2i}}$$

$$=\frac{8-12i-4i+6i^2}{16-4i^2}$$

$$=\frac{8-16i+6(-1)}{16-4(-1)}$$ • Use $i^2=-1$.

$$=\frac{8-16i-6}{16+4}$$

$$=\frac{2-16i}{20}$$

$$=\frac{2}{20}-\frac{16}{20}i=\frac{1}{10}-\frac{4}{5}i$$

Powers of i

Powers of i repeat values in cycles of length 4.

$i^2=-1$ $\quad$ $i^6=-1$
$i^3=-i$ $\quad$ $i^7=-i$
$i^4=1$ $\quad$ $i^8=1$
$i^5=i$ $\quad$ $i^9=i$

Example 36 **Find Powers of i**

Simplify:
a. i^{13}
b. i^{242}
c. i^{2345}

Solution
a. $i^{13}=i\cdot i^{12}=i\cdot\left(\mathbf{i^2}\right)^6=i\cdot(-\mathbf{1})^6=i\cdot(1)=i$
b. $i^{242}=\left(\mathbf{i^2}\right)^{121}=(-\mathbf{1})^{121}=-1$
c. $i^{2345}=i\cdot i^{2344}=i\cdot\left(\mathbf{i^2}\right)^{1172}=i\cdot(-\mathbf{1})^{1172}=i\cdot(1)=i$

SOLVED PROBLEMS

The Real Numbers

In the following set which elements are,

a. Natural numbers
b. Whole numbers
c. Integers
d. Rational numbers
e. Irrational numbers
f. Real numbers

1. $\{-4, -\frac{3}{8}, 0.41, 0, \sqrt{3}, 5, \frac{12}{4}, 4.\overline{31}\}$
Solution
a. $5, \frac{12}{4}$
b. $5, 0, \frac{12}{4}$
c. $-4, 0, \frac{12}{4}, 5$
d. $-4, -\frac{3}{8}, 0.41, 0, 5, \frac{12}{4}, 4.\overline{31}$
e. $\sqrt{3}$
f. all the numbers are real

List the additive inverse and multiplicative inverse of each number.
2. 2
Solution
The additive inverse is -2.
The multiplicative inverse is $\frac{1}{2}$.

3. -7
Solution
The additive inverse is 7.
The multiplicative inverse is $-\frac{1}{7}$.

Find the distance between the points whose coordinates are as follows. Write each answer without the absolute value bars.
4. $-2, 4$
Solution
$d(-2, 4) = |-2 - 4| = |-6| = 6$

5. $-\pi, -1$
Solution
$d(-\pi, -1) = |-\pi - (-1)| = |-\pi + 1|$
Note : $-\pi + 1 < 0$, thus $|-\pi + 1| = -(-\pi + 1)$

$$\begin{aligned} d(-\pi, -1) &= -(-\pi + 1) \\ &= \pi - 1 \end{aligned}$$

6. $-\sqrt{2}, -4$
Solution

$$\begin{aligned} d(-\sqrt{2}, -4) &= |-\sqrt{2} - (-4)| \\ &= |-\sqrt{2} + 4| \end{aligned}$$

Note : $-\sqrt{2} + 4 > 0$, thus $|-\sqrt{2} + 4| = -\sqrt{2} + 4$
$d(-\sqrt{2}, -4) = -\sqrt{2} + 4$

Find the value of each expression.

7. $-|-3|$

Solution

$-|-3| = -3$

8. $-|-5| - |-2|$

Solution

$-|-5| - |-2| = -5 - 2 = -7$

9. $|-2| + |-3| - |-4|$

Solution

$|-2| + |-3| - |-4| = 2 + 3 - 4 = 1$

Name the property of real numbers illustrated in each of the following statements.

10. $\sqrt{2} + 0 = \sqrt{2}$

Solution

Identity property of addition.

11. $1 \cdot \frac{3}{4} = \frac{3}{4}$

Solution

Identity property of multiplication.

12. $2 + (-2) = 0$

Solution

Inverse property of addition.

13. $\frac{1}{3} \cdot (3) = 1$

Solution

Inverse property of multiplication.

Perform the indicated operations. Write each answer in lowest terms.

14. $-\frac{5}{4} \cdot \frac{12}{50}$

Solution

$$-\frac{5}{4} \cdot \frac{12}{50} = -\frac{60}{200} = -\frac{3}{10}$$

15. $-\frac{14}{24} \div \frac{7}{12}$

Solution

$$-\frac{14}{24} \div \frac{7}{12} = -\frac{14}{24} \cdot \frac{12}{7}$$

$$= -1$$

16. $4 \cdot \left(\frac{-9}{12}\right) \cdot \frac{3}{2}$

Solution

$$4 \cdot \left(-\frac{9}{12}\right) \cdot \frac{3}{2} = -\frac{36}{12} \cdot \frac{3}{2}$$

$$= -\frac{9}{2}$$

17. $\dfrac{5 \cdot 6 - 6 \div 2}{5 + 4}$

Solution

$$\frac{5 \cdot 6 - 6 \div 2}{5 + 4} = \frac{30 - 3}{9} = 3$$

Integer Exponents

Evaluate each of the following expressions.

18. $\left(3^3 \cdot 3^2\right)^3$

Solution

$$\left(3^3 \cdot 3^2\right)^3 = \left(3^{3+2}\right)^3 = \left(3^5\right)^3 = 3^{15}$$

19. $\left(\dfrac{2^4}{2^7}\right)^{-1}$

Solution

$$\left(\frac{2^4}{2^7}\right)^{-1} = \left(2^{4-7}\right)^{-1} = \left(2^{-3}\right)^{-1} = 2^3 = 8$$

20. $\left(\dfrac{2^{-1} \cdot 3^3}{5^{-1} \cdot 3^2}\right)^{-1}$

Solution

When simplifying problems of this type, move a factor with negative exponent from the numerator to the denominator, and from the denominator to the numerator, by changing the sign of the exponent. Begin by working on the inside of the parentheses.

$$\left(\frac{2^{-1} \cdot 3}{5^{-1}}\right)^{-1} = \left(\frac{5 \cdot 3}{2}\right)^{-1}$$

$$= \left(\frac{15}{2}\right)^{-1}$$

$$= \frac{2}{15}$$

21. $\dfrac{2^{-3}}{\left(2^2 \cdot 5^{-1}\right)^0}$

Solution

Begin by writing $\left(2^2 \cdot 5^{-1}\right)^0 = 1$. Then rewrite 2^{-3} in the numerator as 2^3 in the denominator.

$$\frac{2^{-3}}{\left(2^2 \cdot 5^{-1}\right)^0} = \frac{2^{-3}}{1}$$

$$= \frac{1}{2^3} = \frac{1}{8}$$

22. $\left(-\dfrac{2}{5}\right)^{-3}$

Solution

Begin by writing the expression with positive exponent. To do so, invert the fraction, and write the expression with positive exponent.

$$\left(-\frac{2}{5}\right)^{-3} = \left(-\frac{5}{2}\right)^{3}$$

$$= -\frac{125}{8}$$

23. $\dfrac{6^{-2} - 2^{-1}}{3^{-1}}$

Solution

$$\frac{6^{-2} - 2^{-1}}{3^{-1}} = \frac{\frac{1}{6^2} - \frac{1}{2}}{\frac{1}{3}}$$

$$= \frac{\frac{1}{36} - \frac{1}{2}}{\frac{1}{3}}$$

$$= \frac{\frac{1-18}{36}}{\frac{1}{3}}$$

$$= \frac{-\frac{17}{36}}{\frac{1}{3}} = \frac{-17 \cdot 3}{36} = -\frac{17}{12}$$

Simplify by using the laws of exponents.

24. $\left(2xy^2z^3\right)^{-1}(4xyz)^3$

Solution

$$\left(2xy^2z^3\right)^{-1}(4xyz)^3 = \frac{1}{2xy^2z^3} \cdot (4xyz)^3$$

$$= \frac{4^3x^3y^3z^3}{2xy^2z^3}$$

$$= 32x^2y$$

25. $\left[\left(x^{-1}y^{-3}\right)^{-4}\right]^{-1}$

Solution

Remove inner parentheses by applying the power rule $(a^m)^n = a^{mn}$.

$$\left[\left(x^{-1}y^{-3}\right)^{-4}\right]^{-1} = \left[x^4y^{12}\right]^{-1}$$

$$= \frac{1}{x^4y^{12}}$$

Scientific Notation

Write each number in scientific notation.

26. 4,345,000

Solution

The decimal point in 4,345,000 is to the right of the last zero. Move the decimal point six places to the left to obtain a number between 1 and 10.

$$4,345,000 = 4.345 \times 10^6$$

27. 0.00095

Solution

$0.00095 = 9.5 \times 10^{-4}$

The decimal point is moved four places to the right to obtain a number between 1 and 10.

Write each number in decimal notation.

28. 2.68×10^4

Solution

To write the number in decimal notation, move the decimal four places to the right.

$2.68 \times 10^4 = 26800$

29. 2.3×10^{-3}

Solution

To write the number in decimal notation, move the decimal three places to the left.

$2.3 \times 10^{-3} = 0.0023$

Rational Exponents and Radicals

Evaluate each expression.

30. $\left(-\frac{1}{8}\right)^{-\frac{2}{3}}$

Solution

Method -1 using radicals:

$$\left(-\frac{1}{8}\right)^{-\frac{2}{3}} = \frac{1}{\left(-\frac{1}{8}\right)^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{-\frac{1}{8}}\right)^2} = \frac{1}{\left(-\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4$$

Method -2 using exponents:

$$\left(-\frac{1}{8}\right)^{-\frac{2}{3}} = \left((-2)^{-3}\right)^{-\frac{2}{3}} = (-2)^2 = 4$$

31. $64^{\frac{2}{6}}$

Solution

$64^{\frac{2}{6}} = 64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

32. $-49^{\frac{3}{2}}$

Solution

$$-49^{\frac{3}{2}} = -\left(49^{\frac{1}{2}}\right)^3 = -\left(\sqrt{49}\right)^3 = -(7)^3 = -343$$

33. $81^{-\frac{3}{4}}$

Solution

$$81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{\left(81^{\frac{1}{4}}\right)^3} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$$

34. $\dfrac{9^{-\frac{3}{4}}}{9^{-\frac{5}{4}}}$

Solution

Begin by writing each factor without negative exponent.

$$\frac{9^{-\frac{3}{4}}}{9^{-\frac{5}{4}}} = \frac{9^{\frac{5}{4}}}{9^{\frac{3}{4}}}$$

$$= 9^{\frac{5}{4}-\frac{3}{4}} \qquad \bullet \text{ Apply } \frac{a^p}{a^q} = a^{p-q}.$$

$= 9^{\frac{2}{4}} = 9^{\frac{1}{2}} = 3$

35. $(-16)^{\frac{1}{4}}$

Solution

$(-16)^{\frac{1}{4}}$ is not a real number.

36. $(4)^{3.5}$

Solution

$(4)^{3.5} = 4^{\frac{35}{10}} = 4^{\frac{7}{2}} = \left(4^{\frac{1}{2}}\right)^7 = 2^7 = 128$

Simplify each expression. Assume all variables are positive real numbers.

37. $(64x^3)^{\frac{1}{6}}$

Solution

$(64x^3)^{\frac{1}{6}} = (64)^{\frac{1}{6}}(x^3)^{\frac{1}{6}}$ • Apply $(ab)^q = a^q b^q$.

$= (2^6)^{\frac{1}{6}}(x^3)^{\frac{1}{6}}$ •Write $64 = 2^6$.

$= 2x^{\frac{1}{2}}$ • Apply $(a^p)^q = a^{pq}$ and simplify.

38. $\left(3x^{\frac{3}{4}}\right)\left(-4x^{-\frac{5}{6}}\right)$

Solution

$\left(3x^{\frac{3}{4}}\right)\left(-4x^{-\frac{5}{6}}\right) = (3)(-4)\,x^{\frac{3}{4}}x^{-\frac{5}{6}}$ • Regroup.

$= -12x^{\frac{3}{4}-\frac{5}{6}}$ • Apply $a^p \cdot a^q = a^{p+q}$.

$= -12x^{\frac{9}{12}-\frac{10}{12}}$

$= -12x^{-\frac{1}{12}}$

$= -\dfrac{12}{x^{\frac{1}{12}}}$ • Apply $a^{-p} = \dfrac{1}{a^p}$.

Simplify each of the following radicals. Assume all variables are positive real numbers.

39. $\sqrt[3]{-27x^3y^6}$

Solution

Begin by expressing each factor as a perfect cube, then simplify.

$\sqrt[3]{-27x^3y^6} = \sqrt[3]{(-3)^3\,x^3\,(y^2)^3}$

$= -3xy^2$ • Simplify using $\sqrt[n]{a^n} = a$.

40. $\sqrt[5]{\sqrt{x^{15}}}$

Solution

$\sqrt[5]{\sqrt{x^{15}}} = \sqrt[10]{x^{15}}$ • Apply $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

$= \sqrt[10]{x^{10}x^5}$ • Express radicand as a perfect 10^{th} power.

$= x\,\sqrt[10]{x^5}$ • Simplify using $\sqrt[n]{a^n} = a$.

$= x x^{\frac{5}{10}}$ • Change to exponential form.

$= x\sqrt{x}$

Perform the indicated operations. Assume all variables are positive real numbers.

41. $2\sqrt{50} - \sqrt{2}$

Solution

$2\sqrt{50} - \sqrt{2} = 2\sqrt{5^2 \cdot 2} - \sqrt{2}$
$= 2(5)\sqrt{2} - \sqrt{2}$
$= 10\sqrt{2} - \sqrt{2}$
$= 9\sqrt{2}$

42. $\sqrt{8x+8} - 2\sqrt{2x+2}$

Solution

$\sqrt{8x+8} - 2\sqrt{2x+2} = \sqrt{4(2x+2)} - 2\sqrt{2x+2}$
$= 2\sqrt{2x+2} - 2\sqrt{2x+2}$
$= 0$

43. $\sqrt{2xy} \cdot \sqrt{3x^2y^3} \cdot \sqrt{6x^3y^4}$

Solution

$\sqrt{2xy} \cdot \sqrt{3x^2y^3} \cdot \sqrt{6x^3y^4} = \sqrt{36x^6y^8}$
$= \sqrt{6^2 \cdot (x^3)^2 \cdot (y^4)^2}$
$= 6x^3y^4$

44. $\left(\sqrt{x-5}+2\right)^2$

Solution

$\left(\sqrt{x-5}+2\right)^2 = x - 5 + 4\sqrt{x-5} + 4$
$= x + 4\sqrt{x-5} - 1$

45. $\sqrt{\dfrac{40x^2y}{30xy}}$

Solution

$\sqrt{\dfrac{40x^2y}{30xy}} = \sqrt{\dfrac{4x}{3}}$ • Simplify the radicand.

$= \dfrac{2\sqrt{x}}{\sqrt{3}}$

$= \dfrac{2\sqrt{x}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ • Rationalize the denominator.

$= \dfrac{2\sqrt{3x}}{3}$

Polynomials

Determine the degree of the following polynomials:

46. $4x^2 - 3xy^7 - 2x^2y^5$

Solution

Degree of the polynomial is 8.

47. $y + 4xy + 3x^4$

Solution

Degree of the polynomial is 4.

48. $6x^2y^5 + 3x^4y^4 + 2x^5y$

Solution
Degree of the polynomial is 8.

Perform the indicated operations and simplify.

49. $(5x^2 - 3x + 2) + (5 - 2x + 2x^2)$
Solution
$(5x^2 - 3x + 2) + (5 - 2x + 2x^2) = 7x^2 - 5x + 7$

50. $(2x^2 - 3x + 4)(3x^3 - x - 2)$
Solution
$(2x^2 - 3x + 4)(3x^3 - x - 2) = 2x^2(3x^3 - x - 2) - 3x(3x^3 - x - 2) + 4(3x^3 - x - 2)$
$= 6x^5 - 2x^3 - 4x^2 - 9x^4 + 3x^2 + 6x + 12x^3 - 4x - 8$
$= 6x^5 - 9x^4 + 10x^3 - x^2 + 2x - 8$

51. $(5x - 2)(2x - 5)$
Solution
$(5x - 2)(2x - 5) = 10x^2 - 25x - 4x + 10$
$= 10x^2 - 29x + 10$

52. $(x - 3)^2$
Solution
$(x - 3)^2 = x^2 - 6x + 9$

53. $(x^2 - 5x)^2 - (x^2 - 2x)^2$
Solution
$(x^2 - 5x)^2 - (x^2 - 2x)^2 = x^4 - 10x^3 + 25x^2 - (x^4 - 4x^3 + 4x^2)$
$= x^4 - 10x^3 + 25x^2 - x^4 + 4x^3 - 4x^2$
$= -6x^3 + 21x^2$

54. $[x + (y - 2)][x - (y - 2)]$
Solution
$[x + (y - 2)][x - (y - 2)] = x^2 - (y - 2)^2$
$= x^2 - (y^2 - 4y + 4)$
$= x^2 - y^2 + 4y - 4$

55. $[(x + 3) - y]^2$
Solution
$[(x + 3) - y]^2 = (x + 3)^2 - 2y(x + 3) + y^2$
$= x^2 + 6x + 9 - 2xy - 6y + y^2$

Factoring Techniques

Factor.

56. $24 - 10x + x^2$
Solution
Rewrite the expression with descending powers of x.
$24 - 10x + x^2 = x^2 - 10x + 24$

Determine all possible factors of 24, then find their sums.

Factors	Sum
$+4, +6$	$+10$
$+3, +8$	$+11$
$+2, +12$	$+14$
$-4, -6$	-10
$-3, -8$	-11
$-2, -12$	-14

In the above list the two numbers whose sum is -10 and whose product is 24, are -4 and -6.

Therefore, we have
$x^2 - 10x + 24 = (x - 4)(x - 6)$

57. $x^2 - 8x - 12$
Solution
Determine all possible factors of -12, then find their sums.

Factors	Sum
$-2, +6$	$+4$
$+2, -6$	-4
$+3, -4$	-1
$-3, +4$	$+1$
$-1, +12$	$+11$
$+1, -12$	-11

It is not possible to find two numbers whose sum is -8 and whose product is -12. Therefore, the trinomial is prime.

58. $4x^2 - 12x + 9$
Solution
The first term is a square, $(2x)^2$, as is the last term, $(3)^2$. Since $2(2x)(3x) = 12x^2$,we can factor as follows:
$4x^2 - 12x + 9 = (2x - 3)^2$

59. $9x^2 + 24x + 16$
Solution
The first term is a square, $(3x)^2$, as is the last term, $(4)^2$. Since $2(3x)(4) = 24x$, we can factor as follows:
$9x^2 + 24x + 16 = (3x + 4)^2$

60. $x^2 - 25$
Solution
$x^2 - 25 = (x - 5)(x + 5)$

61. $36x^2 - y^2$
Solution
$36x^2 - y^2 = (6x - y)(6x + y)$

62. $-x^8 + x^4$
Solution
Factor out the common factor, $-x^4$.
$-x^8 + x^4 = -x^4(x^4 - 1)$

$= -x^4\left(x^2-1\right)\left(x^2+1\right)$
$= -x^4\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$
Therefore,
$-x^8+x^4 = -x^4\left(x-1\right)\left(x+1\right).\left(x^2+1\right)$

63. x^5-x^3
Solution
Factor out the common factor, x^3.
$x^5-x^3 = x^3\left(x^2-1\right)$
$= x^3\left(x-1\right)\left(x+1\right)$
Therefore,
$x^5-x^3 = x^3\left(x-1\right)\left(x+1\right)$

64. x^2-6x+9
Solution
$x^2-6x+9=\left(x-3\right)^2$

65. $9x^2-30x+25$
Solution
Notice the perfect square polynomial form, as
First term, $9x^2=\left(3x\right)^2$
Middle term, $-30x=-2\left(3x\right)\left(5\right)$
Last term, $25=\left(5\right)^2$
Thus,
$9x^2-30x+25=\left(3x-5\right)^2$

66. $25x^2+30x+9$
Solution
Recognize the perfect square polynomial form, as
First term, $25x^2=\left(5x\right)^2$
Middle term, $30x=2\left(5x\right)\left(3\right)$
Last term, $9=\left(3\right)^2$
Thus,
$25x^2+30x+9=\left(5x+3\right)^2$

67. $x^2+12x+36$
Solution
Notice the perfect square polynomial form, as
First term, $x^2=\left(x\right)^2$
Middle term, $12x=2\left(x\right)\left(6\right)$
Last term, $36=\left(6\right)^2$
Thus,
$x^2+12x+36=\left(x+6\right)^2$

68. $27x^3+1$
Solution
Express the binomial $27x^3+1$ as the sum of two cubes. Hence, we have

$$\begin{aligned}27x^3+1 &= \left(3x\right)^3+\left(1\right)^3\\ &= \left(3x+1\right)\left(9x^2-3x+1\right)\end{aligned}$$

69. x^3-64
Solution

$$\begin{aligned}x^3-64 &= \left(x\right)^3-\left(4\right)^3\\ &= \left(x-4\right)\left(x^2+4x+16\right)\end{aligned}$$

70. $x^6 - 64$

Solution

$$\begin{aligned} x^6 - 64 &= \left(x^2\right)^3 - (4)^3 \\ &= \left(x^2 - 4\right)\left(x^4 + 4x^2 + 16\right) \\ &= (x+2)(x-2)\left(x^4 + 4x^2 + 16\right) \end{aligned}$$

71. $8x^3 + 27y^3$

Solution

$$\begin{aligned} 8x^3 + 27y^3 &= (2x)^3 + (3y)^3 \\ &= (2x + 3y)\left(4x^2 - 6xy + 9y^2\right) \end{aligned}$$

72. $5x^2 - 15x - 15y + 5xy$

Solution

Factor by grouping.

$$\begin{aligned} 5x^2 - 15x - 15y + 5xy &= 5\left(x^2 - 3x - 3y + xy\right) \\ &= 5\left[x(x-3) + y(x-3)\right] \\ &= 5(x-3)(x+y) \end{aligned}$$

73. $9x^4 + 3x^3 - 36x^2 - 12x$

Solution

First factor $3x$ from all terms, then see if the remaining four terms within parentheses can be factored by grouping.

$$\begin{aligned} 9x^4 + 3x^3 - 36x^2 - 12x &= 3x\left(3x^3 + x^2 - 12x - 4\right) \\ &= 3x\left[x^2(3x+1) - 4(3x+1)\right] \\ &= 3x(3x+1)\left(x^2 - 4\right) \\ &= 3x(3x+1)(x-2)(x+2) \end{aligned}$$

74. $a^2 + 2a + 1 - b^2 - 4b - 4$

Solution

$$\begin{aligned} a^2 + 2a + 1 - b^2 - 4b - 4 &= \left(a^2 + 2a + 1\right) - \left(b^2 + 4b + 4\right) \\ &= (a+1)^2 - (b+2)^2 \\ &= [(a+1) - (b+2)]\,[(a+1) + (b+2)] \\ &= [a - b - 1]\,[a + b + 3] \end{aligned}$$

Rational Expressions

Find the domain of each expression.

75. $\dfrac{3}{x^2 + 5x}$

Solution

Set the denominator equal to zero and solve for x.

$x^2 + 5x = 0$

$x(x+5) = 0$

$x = 0 \quad \text{or} \quad x = -5$

$\text{Domain} = \{x \mid x \neq 0 \quad \text{or} \quad x \neq -5\}.$

76. $\dfrac{x+4}{x^2 - 25}$

Solution

Set the denominator equal to zero and solve for x.

$x^2 - 25 = 0$

$(x-5)(x+5) = 0$

$x = 5 \quad \text{or} \quad x = -5$

$\text{Domain} = \{x \mid x \neq 5 \quad \text{or} \quad x \neq -5\}.$

Simplify each rational expression.

77. $\dfrac{2x+4}{x^2-4}$

Solution

$$\frac{2x+4}{x^2-4} = \frac{2(x+2)}{(x-2)(x+2)}$$ • Factor.

$$= \frac{2}{x-2}$$ • Simplify.

78. $\dfrac{4x-8}{x^2-2x}$

Solution

$$\frac{4x-8}{x^2-2x} = \frac{4(x-2)}{x(x-2)}$$ • Factor.

$$= \frac{4}{x}$$ • Simplify.

79. $\dfrac{2x^3}{6x^4+10x^3}$

Solution

$$\frac{2x^3}{6x^4+10x^3} = \frac{2x^3}{2x^3(3x+5)}$$ • Factor the denominator.

$$= \frac{1}{3x+5}$$ • Simplify.

80. $\dfrac{x^3+x^2-6x}{x^2-2x}$

Solution

$$\frac{x^3+x^2-6x}{x^2-2x} = \frac{x(x^2+x-6)}{x(x-2)}$$ • Factor out the GCF.

$$= \frac{x(x+3)(x-2)}{x(x-2)}$$ • Factor completely.

$$= x+3$$ • Simplify.

81. $\dfrac{x^3+4x^2-21x}{(x+2)^2-25}$

Solution

$$\frac{x^3+4x^2-21x}{(x+2)^2-25} = \frac{x(x^2+4x-21)}{[(x+2)-5][(x+2)+5]}$$

$$= \frac{x(x+7)(x-3)}{(x-3)(x+7)}$$

$$= x$$

82. $\dfrac{3}{3-2x} + \dfrac{2x}{2x-3}$

Solution

$$\frac{3}{3-2x} + \frac{2x}{2x-3} = \frac{3}{-(2x-3)} + \frac{2x}{2x-3}$$ • Use $3-2x = -(2x-3)$.

$$= \frac{-3}{2x-3} + \frac{2x}{2x-3}$$

$$= \frac{-3+2x}{2x-3}$$

$$= \frac{2x-3}{2x-3}$$ • Rearrange the numerator.

$$= 1$$

83. $\dfrac{2}{x^2-3x+2} + \dfrac{3}{x^2-4x+3} - \dfrac{5}{x^2-5x+6}$

Solution

Since $x^2-3x+2 = (x-2)(x-1)$, $x^2-4x+3 = (x-3)(x-1)$, and $x^2-5x+6 = (x-2)(x-3)$, the LCD is $(x-2)(x-1)(x-3)$.

$$\frac{2}{x^2-3x+2} + \frac{3}{x^2-4x+3} - \frac{5}{x^2-5x+6}$$

$$= \frac{2}{(x-2)(x-1)} + \frac{3}{(x-3)(x-1)} - \frac{5}{(x-2)(x-3)}$$

$$= \frac{\mathbf{x-3}}{\mathbf{x-3}} \cdot \frac{2}{(x-2)(x-1)} + \frac{\mathbf{x-2}}{\mathbf{x-2}} \cdot \frac{3}{(x-3)(x-1)} - \frac{\mathbf{x-1}}{\mathbf{x-1}} \cdot \frac{5}{(x-2)(x-3)}$$

$$= \frac{2(x-3)+3(x-2)-5(x-1)}{(x-3)(x-2)(x-1)}$$

$$= \frac{2x-6+3x-6-5x+5}{(x-3)(x-2)(x-1)}$$

$$= \frac{-7}{(x-3)(x-2)(x-1)}$$

84. $\dfrac{x^2+2}{x^3-1} - \dfrac{1}{x-1}$

Solution

Since $x^3-1 = (x-1)\left(x^2+x+1\right)$, the LCD is $\left(x^2+x+1\right)(x-1)$.

$$\frac{x^2+2}{x^3-1} - \frac{1}{x-1} = \frac{x^2+2}{(x-1)(x^2+x+1)} - \frac{1}{x-1}$$

$$= \frac{x^2+2}{(x-1)(x^2+x+1)} - \frac{\mathbf{x^2+x+1}}{\mathbf{x^2+x+1}} \cdot \frac{1}{x-1}$$

$$= \frac{x^2+2-x^2-x-1}{(x-1)(x^2+x+1)}$$

$$= \frac{1-x}{(x-1)(x^2+x+1)}$$ • Simplify the numerator.

$$= \frac{-(x-1)}{(x-1)(x^2+x+1)}$$ • Replace $(1-x)$ by $-(x-1)$.

$$= \frac{-1}{(x^2+x+1)}$$ • Simplify.

85. $\dfrac{x^2-4}{x^2+x-6}\cdot\dfrac{x^3+2x^2-3x}{x^2+2x}$

Solution

First, factor the numerators and denominators completely, then simplify and write the answer in lowest terms.

$$\frac{x^2-4}{x^2+x-6}\cdot\frac{x^3+2x^2-3x}{x^2+2x}=\frac{(x-2)(x+2)}{(x+3)(x-2)}\cdot\frac{x(x^2+2x-3)}{x(x+2)}$$

$$=\frac{1}{x+3}\cdot\frac{x(x+3)(x-1)}{x}$$

$$=x-1$$

86. $\dfrac{x^2}{(2x-y)^2}\cdot\dfrac{4x^2-y^2}{2x^2+xy}$

Solution

$$\frac{x^2}{(2x-y)^2}\cdot\frac{4x^2-y^2}{2x^2+xy}=\frac{x^2}{(2x-y)^2}\cdot\frac{(2x+y)(2x-y)}{x(2x+y)}$$

$$=\frac{x}{2x-y}$$

87. $\dfrac{2x^2+15x+27}{x^2-2x+4}\cdot\dfrac{x^2-x-6}{x^2-9}\div\dfrac{4x+18}{x^3+8}$

Solution

Begin by inverting the divisor. Then factor each numerator and denominator completely.

$$\frac{2x^2+15x+27}{x^2-2x+4}\cdot\frac{x^2-x-6}{x^2-9}\div\frac{4x+18}{x^3+8}=\frac{2x^2+15x+27}{x^2-2x+4}\cdot\frac{x^2-x-6}{x^2-9}\cdot\frac{x^3+8}{4x+18}$$

$$=\frac{(2x+9)(x+3)}{x^2-2x+4}\cdot\frac{(x-3)(x+2)}{(x-3)(x+3)}\cdot\frac{(x+2)(x^2-2x+4)}{2(2x+9)}$$

$$=\frac{(x+2)^2}{2}$$

- Divide out the common factors.

88. $\dfrac{8x^2+2x-15}{-2x^3-3x^2}\div\dfrac{8x^2-14x+5}{1-4x^2}$

Solution

First, invert the divisor. Next, factor each rational expression completely.

$$\frac{8x^2+2x-15}{-2x^3-3x^2}\div\frac{8x^2-14x+5}{1-4x^2}=\frac{8x^2+2x-15}{-2x^3-3x^2}\cdot\frac{1-4x^2}{8x^2-14x+5}$$

$$=\frac{(2x+3)(4x-5)}{-x^2(2x+3)}\cdot\frac{(1-2x)(1+2x)}{(2x-1)(4x-5)}$$

$$=\frac{1}{-x^2}\cdot\frac{(1-2x)(1+2x)}{2x-1}$$

- Divide out common factors.

$$= \frac{1}{-x^2} \cdot \frac{(1-2x)(1+2x)}{-(1-2x)}$$

- Use $(2x-1) = -(1-2x)$.

$$= \frac{1+2x}{x^2}$$

- Divide out common factors.

89. $\dfrac{2+\dfrac{1}{x}}{4-\dfrac{1}{x}}$

Solution

Simplify the numerator and denominator of the rational expression by combining terms.

$$\frac{2+\dfrac{1}{x}}{4-\dfrac{1}{x}} = \frac{\dfrac{\mathrm{x}}{\mathrm{x}}\cdot\dfrac{2}{1}+\dfrac{1}{x}}{\dfrac{\mathrm{x}}{\mathrm{x}}\cdot\dfrac{4}{1}-\dfrac{1}{x}}$$

$$= \frac{\dfrac{2x+1}{x}}{\dfrac{4x-1}{x}}$$

$$= \frac{x(2x+1)}{x(4x-1)}$$

- Perform division.

$$= \frac{2x+1}{4x-1}$$

- Divide out common factor.

90. $\dfrac{3-\dfrac{6}{x+1}}{\dfrac{x(x+1)}{x^2+2x+1}}$

Solution

$$\frac{3-\dfrac{6}{x+1}}{\dfrac{x(x+1)}{x^2+2x+1}}$$

$$= \frac{\dfrac{\mathbf{x+1}}{\mathbf{x+1}}\cdot 3-\dfrac{6}{x+1}}{\dfrac{x(x+1)}{(x+1)^2}}$$

- Simplify the numerator and factor denominator.

$$= \frac{\dfrac{3x+3-6}{x+1}}{\dfrac{x(x+1)}{(x+1)^2}}$$

$$= \frac{\dfrac{3x-3}{x+1}}{\dfrac{x(x+1)}{(x+1)^2}}$$

$$= \frac{3(x-1)(x+1)^2}{x(x+1)(x+1)}$$

- Perform division.

$$= \frac{3(x-1)}{x}$$

- Reduce to lowest terms.

91. $\dfrac{x^{-1}y - xy^{-1}}{x^2 - y^2}$

Solution

$$\frac{x^{-1}y - xy^{-1}}{x^2 - y^2} = \frac{\dfrac{y}{x} - \dfrac{x}{y}}{x^2 - y^2}$$

- Eliminate negative exponents.

$$= \frac{\dfrac{\mathrm{y}}{\mathrm{y}} \cdot \dfrac{y}{x} - \dfrac{\mathrm{x}}{\mathrm{x}} \cdot \dfrac{x}{y}}{x^2 - y^2}$$

- Simplify the numerator.

$$= \frac{\dfrac{y^2}{xy} - \dfrac{x^2}{xy}}{x^2 - y^2}$$

$$= \frac{\dfrac{y^2 - x^2}{xy}}{x^2 - y^2}$$

$$= \frac{\dfrac{-(x^2 - y^2)}{xy}}{x^2 - y^2}$$

- Use $y^2 - x^2 = -(x^2 - y^2)$.

$$= \frac{-(x^2 - y^2)}{xy(x^2 - y^2)}$$

- Perform division.

$$= \frac{-1}{xy}$$

- Reduce to lowest terms.

Complex Numbers

Perform the indicated operations and write the result in standard form.

92. $(5 + 2i) + (7 + 3i)$

Solution

$$\begin{aligned}(5 + 2i) + (7 + 3i) &= (5 + 7) + (2 + 3)i \\ &= 12 + 5i\end{aligned}$$

93. $(-2 - i)(-3 + 2i)$

Solution

$$\begin{aligned}(-2 - i)(-3 + 2i) &= 6 - 4i + 3i - 2i^2 \\ &= 6 - i - 2(-1) \\ &= 6 - i + 2 \\ &= 8 - i\end{aligned}$$

94. $(5-3i)(5+3i)$

Solution

$$\begin{aligned}(5-3i)(5+3i) &= (5)^2-(3i)^2\\ &= 25-9i^2\\ &= 25-9(-1)\\ &= 25+9\\ &= 34\end{aligned}$$

95. $(2-5i)^2$

Solution

$$\begin{aligned}(2-5i)^2 &= (2)^2-2(2)(5i)+(5i)^2 && \bullet\ \text{Use } (a-b)^2=a^2-2ab+b^2.\\ &= 4-20i+25i^2\\ &= 4-20i+25(-1) && \bullet\ \text{Use } i^2=-1.\\ &= 4-25-20i\\ &= -21-20i\end{aligned}$$

96. $\dfrac{5-2i}{4+i}$

Solution

$$\begin{aligned}\frac{5-2i}{4+i} &= \frac{(5-2i)}{(4+i)}\cdot\frac{(4-i)}{(4-i)} && \bullet\ \text{Rationalize the denominator.}\\ &= \frac{20-5i-8i+2i^2}{16-i^2}\\ &= \frac{20-13i+2(-1)}{16-(-1)} && \bullet\ \text{Replace } i^2 \text{ with } -1.\\ &= \frac{20-13i-2}{16+1}\\ &= \frac{18-13i}{17}\\ &= \frac{18}{17}-\frac{13}{17}i && \bullet\ \text{Write answer in standard form}\end{aligned}$$

Simplify.

97. i^{718}

Solution

$$\begin{aligned}i^{718} &= \left(i^2\right)^{359}\\ &= (-1)^{359}\\ &= -1\end{aligned}$$

98. i^{463}

Solution

$$\begin{aligned}i^{463} &= i\cdot i^{462}\\ &= i\cdot\left(i^2\right)^{231}\\ &= i\cdot(-1)^{231}\\ &= -i\end{aligned}$$

99. $\dfrac{1}{i^{103}}$

Solution

$$\frac{1}{i^{103}}=\frac{1}{i\cdot i^{102}}$$

$= \dfrac{1}{i \cdot (i^2)^{51}}$

$= \dfrac{1}{i \cdot (-1)^{51}}$ • Use $i^2 = -1$.

$= \dfrac{1}{-i}$

$= \dfrac{1}{-i} \cdot \dfrac{\mathbf{i}}{\mathbf{i}}$ • Rationalize the denominator

$= \dfrac{i}{-i^2}$

$= \dfrac{i}{-(-1)}$ • Use $i^2 = -1$.

$= i$

Evaluate each expression and write the result in standard form.

100. $\sqrt{-2} \cdot \sqrt{-2}$

Solution

$\sqrt{-2} \cdot \sqrt{-2} = i\sqrt{2} \cdot i\sqrt{2}$ • Use $\sqrt{-a} = i\sqrt{a}$.

$= i^2\sqrt{4}$

$= (-1)(2)$ • Use $i^2 = -1$.

$= -2$

101. $\dfrac{\sqrt{-3}}{\sqrt{-5}}$

Solution

$\dfrac{\sqrt{-3}}{\sqrt{-5}} = \dfrac{i\sqrt{3}}{i\sqrt{5}}$ • Use $\sqrt{-a} = i\sqrt{a}$.

$= \dfrac{\sqrt{3}}{\sqrt{5}}$

$= \dfrac{\sqrt{3}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$ • Rationalize the denominator.

$= \dfrac{\sqrt{15}}{5}$

102. $\left(2+\sqrt{-3}\right)\left(3-\sqrt{-2}\right)$

Solution

$\left(2+\sqrt{-3}\right)\left(3-\sqrt{-2}\right) = \left(2+i\sqrt{3}\right)\left(3-i\sqrt{2}\right)$

$= 6 - 2i\sqrt{2} + 3i\sqrt{3} - i^2\sqrt{6}$

$= 6 - 2i\sqrt{2} + 3i\sqrt{3} + \sqrt{6}$ • Use $i^2 = -1$.

$= \left(6+\sqrt{6}\right) + \left(3\sqrt{3} - 2\sqrt{2}\right) i$

103. $\left(5-\sqrt{-2}\right)^2$

Solution

$\left(5-\sqrt{-2}\right)^2 = \left(5-i\sqrt{2}\right)^2$

$= 25 - 10i\sqrt{2} + 2i^2$

$= 25 - 10i\sqrt{2} - 2$ • Use $i^2 = -1$.

$= 23 - 10i\sqrt{2}$

PRACTICE TEST

Evaluate.

1. $-64^{\frac{2}{3}}$

2. $5^{\frac{1}{4}} \cdot 5^{-\frac{1}{2}}$

3. $2\left(3x-5\right)^0$

Simplify. Write answer with positive exponent.

4. $\left(\dfrac{x^4y^{-6}}{xy^{-3}}\right)^{\frac{1}{3}}$

5. $\dfrac{x^{\frac{1}{3}}y^{\frac{2}{3}}}{x^{\frac{1}{2}}y^{\frac{1}{2}}}$

6. $\left(\dfrac{-2x^2y^{-3}}{3xy^{-2}}\right)^{-2}$

7. $\dfrac{-25x^3y^5}{5x^8y^2}$

8. $\sqrt{162x^2} - x\sqrt{128}$

9. $\sqrt[3]{81x^6y^{12}}$

10. $\dfrac{3x}{\sqrt[3]{2y}}$

11. $\sqrt{\dfrac{25x^4}{2y}}$

12. $\sqrt{32x^2y} \cdot \sqrt{2x^4y^3}$

13. $\left(\sqrt{x+1}-1\right)^2$

Simplify each expression. Write answer in standard form.

14. $\left(1-\sqrt{-2}\right)\left(2-\sqrt{-3}\right)$

15. $(2i+3)-(i-1)$

16. $(4 - 8i)(2 + 5i)$

17. $\dfrac{6i}{2 - 3i}$

18. i^{653}

Simplify.

19. $\dfrac{6y^2 + 12y + 6}{y^3 + y^2 - y - 1}$

20. $\dfrac{3x^2 - 17x - 6}{x^2 - 1} \div \dfrac{x - 6}{x - 1}$

21. $\dfrac{-4y}{y^2-1}+\dfrac{4}{y-1}$

22. $\dfrac{1-\frac{5}{y}}{3+\frac{2}{y}}$

CHAPTER TWO

Equations and Inequalities

- 2.1 Linear Equations
- 2.2 Applications
- 2.3 Quadratic Equations
- 2.4 Other Types of Equations
- 2.5 Inequalities
- 2.6 Absolute Value Equations and Inequalities

2.1 LINEAR EQUATIONS

A **linear equation** in one variable is an equation written in the form

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$.

Steps that result in Equivalent Equations

- Add or subtract the same expression from both sides of the equation.
  $x = a, x + b = a + b$, and $x - b = a - b$ are equivalent.
- Interchange left-hand and right-hand sides of the equation.
  $a = x$ is equivalent to $x = a$.
- Divide both sides of the equation by the same nonzero expression.
  $\frac{x}{c} = \frac{a}{c}, c \neq 0$ is equivalent to $x = a$.
- Multiply both sides of the equation by the same nonzero expression.
  $xc = ac, c \neq 0$ is equivalent to $x = a$.
- Simplify each side of the equation by combining like terms.

Steps for Solving Linear Equations

1. Eliminate any fractions by multiplying both sides of the equation by the LCD.
2. Remove all parentheses by using distributive property.
3. Simplify each side of the equation by combining like terms.
4. Isolate the variable on one side of the equation.
5. Make the coefficient of the variable one, by dividing both sides of the equation by that coefficient.
6. Find the domain of the variable. If the solution is not in the domain of the variable, then there is no solution.

Example 1 **Solve a Linear Equation**

Solve: $\frac{x+3}{6} - \frac{7}{9} = \frac{x-3}{9}$

Solution

$$\frac{x+3}{6} - \frac{7}{9} = \frac{x-3}{9}$$

$$18\left(\frac{x+3}{6} - \frac{7}{9}\right) = 18\left(\frac{x-3}{9}\right)$$ • Multiply each side by 18.

$$3(x+3) - 2(7) = 2(x-3)$$
$$3x + 9 - 14 = 2x - 6$$
$$3x - 2x = -6 + 14 - 9$$ • Collect like terms.
$$x = -1$$

The solution set is $\{-1\}$.

Example 2 **Solve a Linear Equation**

Solve: $5(4x+1) - 65 = 4(2x-3)$

Solution

$5(4x+1) - 65 = 4(2x-3)$
$20x + 5 - 65 = 8x - 12$ • use the distributive property to remove parentheses.

$$\begin{aligned} 20x - 8x &= 65 - 5 - 12 && \bullet \text{ Collect like terms.} \\ 12x &= 48 \\ x &= 4 && \bullet \text{ Divide each side by 12.} \end{aligned}$$

The solution set is $\{4\}$.

CAUTION: It is important to check proposed solutions whenever each side of the equation is multiplied by a variable expression.

Example 3 **Solve a Linear Equation that has No Solution**

Solve: $\dfrac{1}{x-5} + \dfrac{10}{x+5} = \dfrac{10}{x^2-25}$

Solution

$$\frac{1}{x-5} + \frac{10}{x+5} = \frac{10}{x^2-25}$$

$$\frac{1}{x-5} + \frac{10}{x+5} = \frac{10}{(x-5)(x+5)} \qquad \bullet \text{ Factor using } a^2 - b^2 = (a-b)(a+b)$$

Next, multiply each side by $(x-5)(x+5)$, the LCD.

$$(\mathrm{x}-5)(\mathrm{x}+5)\left[\frac{1}{x-5} + \frac{10}{x+5}\right] = (\mathrm{x}-5)(\mathrm{x}+5)\left[\frac{10}{(x-5)(x+5)}\right]$$

$$\begin{aligned} x + 5 + 10(x-5) &= 10 \\ x + 5 + 10x - 50 &= 10 \\ 11x - 45 &= 10 \\ 11x &= 55 \\ x &= 5 \end{aligned}$$

The proposed solution, 5, is not a solution because substitution of it in the original equation results in denominator of zero. Thus, the solution set contains no element and is written as $\{\}$, or $\emptyset$.

Literal Equations

A **literal equation** is an equation that involves more than one variable. An equation of this type can be solved for a specified variable in terms of others, by treating the other variables as constants.

Steps For Solving Literal Equations

1. Clear all fractions by multiplying both sides of the equation by the least common denominator.

2. Remove all parentheses and simplify.

3. Collect all terms containing the *specified variable* on the left side of the equation and all the other terms on the right side of the equation.

4. Factor out the *specified variable.*

5. Solve for the *specified variable* by dividing both sides of the equation by the coefficient of the *specified variable.*

Example 4 **Solve a Literal Equation**

Solve $A = \dfrac{h}{2}(a+b)$ for b.

Solution

$$A = \frac{h}{2}(a+b)$$

$$\mathbf{2}(A) = \mathbf{2}\left[\frac{h}{2}(a+b)\right]$$

- Multiply each side by 2, the LCD.

$$2A = ha + hb$$
$$2A - \mathbf{ha} = ha + hb - \mathbf{ha}$$
$$2A - ha = hb$$

- Subtract ha from each side.

$$b = \frac{2A - ha}{h}$$

- Divide each side by h.

2.2 APPLICATIONS

Strategies for Solving Word Problems

1. Read the problem carefully and determine what is known and what must be found.

2. Assign letter x to one of the unknowns and express the other unknown in terms of x.

3. Use the information from step 2 together with a known formula or some additional information given in the problem, to write an equation.

4. Solve the equation in step 3.

5. Check the answer with facts given in the problem and make certain that it satisfies the conditions of the problem.

Here are some tips:

Integer Problems

In solving integer problems, it will help to know that:

Three consecutive integers are:	$x, x+1, x+2$
Three consecutive even integers are:	$x, x+2, x+4$
Three consecutive odd integers are:	$x, x+2, x+4$

Example 5 **Solve an Integer Problem**

Find three consecutive even integers whose sum is 348.

Solution

Let $x, x+2$, and $x+4$ represent the three consecutive even integers. Thus,

$$\begin{aligned} x+(x+2)+(x+4) &= 348 \\ 3x+6 &= 348 \\ 3x &= 342 \\ x &= 114 \end{aligned}$$

This gives $x+2 = 116$ and $x+4 = 118$. Therefore, the three consecutive even integers are 114, 116, and 118.

Investment Problems

To solve simple interest problems use the equation $I = prt$; where

- I is the simple interest
- p is the principal
- r is the interest rate
- t is the time in years.

Example 6 **Solve an Investment Problem**

Rob invested \$35,000 into two certificate of deposit accounts. One investment earned 6% interest and the other 8%. The amount of interest for one year was the same for both accounts. How much has he invested in each account?

Solution

Let x = amount invested at 6%

$35{,}000 - x$ = amount invested at 8%

Using $I = prt$, with $t = 1$ year gives

Investment	**principal**	×	**rate**	×	**time**	=	**interest**
1	x		0.06		1		$0.06x$
2	$35{,}000 - x$		0.08		1		$0.08\,(35{,}000 - x)$

The expressions in the interest column must be equal, since the interest earned for both accounts were the same. Therefore,

$$\begin{aligned} 0.06x &= 0.08\,(35,000 - x) \\ 0.06x &= 2800 - 0.08x \\ 0.06x + 0.08x &= 2800 \\ 0.14x &= 2800 \\ x &= \$20,000 \quad @\,6\% \\ 35,000 - x &= \$15,000 \quad @\,8\% \end{aligned}$$

Work Problems

To solve work problems use the equation

rate of work × time worked = part of task completed

Note: rate of work is the portion of task completed in one unit of time.

Example 7 Solve a Work Problem

John can paint a room in 3 hours, Chris can paint the same room in 2 hours. How long would it take them working together to paint the room?

Solution

John can paint a room in 3 hours, then in one hour he can paint $\frac{1}{3}$ of the room. Thus, his rate of work is $\frac{1}{3}$ of the room per hour. In the same manner Chris' rate of work is $\frac{1}{2}$.

Let $t =$ the number of hours it takes both to paint the room together.

	rate	×	**time**	=	**Part of job completed**
John	$\frac{1}{3}$		t		$\frac{t}{3}$
Chris	$\frac{1}{2}$		t		$\frac{t}{2}$

$$\begin{pmatrix} \textit{Part of job completed} \\ \textit{by John} \\ \textit{when working with} \\ \textit{Chris} \end{pmatrix} + \begin{pmatrix} \textit{Part of job completed} \\ \textit{by Chris} \\ \textit{when working with} \\ \textit{John} \end{pmatrix} = (\textit{One painted room})$$

$$\frac{t}{3} + \frac{t}{2} = 1$$

$$\mathbf{6}\left(\frac{t}{3} + \frac{t}{2}\right) = \mathbf{6}\,(1)$$

$$\begin{aligned} 2t + 3t &= 6 \\ 5t &= 6 \end{aligned}$$

$$t = \frac{6}{5} = 1.2$$

Therefore, it takes them 1 hour and 12 minutes to paint the room together.

Uniform Motion Problems

To solve uniform motion problems use the equation

$d = rt,$

where

- $r =$ *the rate of speed*

- *t = the time*
- *d = the distance travelled.*

Example 8 **Solve a Uniform Motion Problem**

Two planes leave Chicago at the same time travelling in the same direction. One plane travels at 480 miles per hour, and the second travels at 360 miles per hour. In how many hours will they be 840 miles apart?

Solution

Let t = number of hours needed for planes to be 840 miles apart.

	rate	×	time	=	distance
Plane 1	480		t		$480t$
Plane 2	360		t		$360t$

$480t - 360t = 840$
$480t - 360t = 840$
$120t = 840$
$t = 7$

Therefore, in 7 hours the planes will be at a distance of 840 miles.

Value Mixture Problems

A **value mixture problem** involves combining two quantities that have different prices to produce different ingredient. For solving value mixture problems use

$AP = V$, where

- A is amount of the ingredient
- P is the price per unit of the ingredient
- V is the value of the ingredient

Example 9 **Solve a Value Mixture Problem**

A coffee shop owner wishes to mix a coffee that sells for \$6 per pound with a coffee that sells for \$10 per pound to obtain 30 pounds of a new blend that is worth \$8 per pound. How many pounds of each type of coffee should he use?

Solution

Let x = pounds of \$6 coffee
$30 - x$ = pounds of \$10 coffee

	Amount	×	Price	=	Value
Type-1	x		6		$6x$
Type-2	$30 - x$		10		$10(30 - x)$
Mixture	30		8		$30(8)$

The sum of the values before mixing is equal to the value after mixing.

$6x + 10(30 - x) = 8(30)$
$6x + 300 - 10x = 240$
$-4x = -60$
$x = 15$ lb \$6 coffee
$30 - x = 15$ lb \$10 coffee

Percent Mixture Problems

A **percent mixture problem** involves combining two quantities that have different strengths to produce a quantity with a different strength. For solving percent mixture problems use $Ar = Q$, where

- A is the amount of the solution or alloy
- r is the strength of solution
- Q is quantity of a substance in the solution or alloy

Example 10 **Solve a Percent Mixture Problem**

Suppose 10 liters of hydrochloric acid solution is mixed with 4 liters of a 30% hydrochloric acid solution to produce a 20% acid solution. What is the strength of the first acid?

Solution

Let $x =$ strength of the acid

	Amount	×	**Strength of acid**	=	**Amount of acid**
Acid 1	10		x		$10x$
Acid 2	4		0.30		$4\,(0.30)$
Mixture	14		0.20		$14\,(0.20)$

$$\begin{pmatrix} \textit{liters of acid} \\ \textit{in Acid 1} \end{pmatrix} + \begin{pmatrix} \textit{liters of acid} \\ \textit{in Acid 2} \end{pmatrix} = \begin{pmatrix} \textit{liters of acid} \\ \textit{Mixture} \end{pmatrix}$$

$$10x \quad + \quad 4\,(0.30) \quad = \quad 14\,(0.20)$$

$$10x + 1.2 = 2.8$$

$$10x = 1.6$$

$$x = 0.16$$

The strength of the first acid is 16%.

Business problems

To solve business problems use

- *profit* = *revenue* − *cost*
- *total revenue* = (*price per unit*) (*number of units*)
- *total cost* = *fixed cost* + *variable cost*

Example 11 **Solve a Business Problem**

A publisher sells a physics book for \$65.20. Suppose the cost per unit is \$40.70, how many books must the publisher sell to have a profit of \$12, 250?

Solution

Let $x =$ number of units of books sold

$$\textit{Profit} = \textit{Revenue} - \textit{Cost}$$

$$12,250 = 65.20x - 40.70x$$

$$12,250 = 24.50x$$

$$x = 500$$

The publisher must sell 500 copies of the book.

2.3 QUADRATIC EQUATIONS

A **quadratic equation** in one variable is an equation written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

METHODS FOR SOLVING QUADRATIC EQUATIONS

Zero Product Property

If X and Y are algebraic expressions, then

$$XY = 0 \qquad \text{if and only if} \qquad X = 0 \text{ or } Y = 0.$$

Example 12 **Solve by Using the Zero Product Property**

Solve by using the zero product property.

a. $2x^2 - 7x = 15$

b. $6x^2 - 3x = 0$

Solution

a. First write the equation in standard form as

$$2x^2 - 7x - 15 = 0$$

Now factor the equation to get

$$(2x + 3)(x - 5) = 0$$

By the zero product property, the product $(2x + 3)(x - 5)$ can only equal zero if

$$2x + 3 = 0 \qquad \text{or} \qquad x - 5 = 0.$$

Solve each of these equations separately to find that the solutions are $-\frac{3}{2}$ and 5. Check these solutions in the original equation. The solution set is $\left\{-\frac{3}{2}, 5\right\}$.

b. The equation is written in standard form. Factor the equation to get

$$6x^2 - 3x = 0$$
$$3x(2x - 1) = 0$$

Next, set each factor equal to zero.

$$3x = 0 \qquad \text{or} \qquad (2x - 1) = 0$$

Solve each equation separately to get 0 and $\frac{1}{2}$. A check will show that these are the solutions of the given equation. The solution set is $\{0, \frac{1}{2}\}$.

The Square Root Property

If X and Y are algebraic expressions such that

$$X^2 = Y \qquad \text{then} \qquad X = \pm\sqrt{Y}$$

Example 13 **Solve by Using the Square Root Property**

Solve by the square root method.

a. $(x - 1)^2 = 5$

b. $(x + 2)^2 = (2x + 3)^2$

Solution

a. Apply the square root property.

$$(x - 1) = \pm\sqrt{5}$$

Next, solve for x.

$$x = 1 \pm \sqrt{5}$$

The solution set is $\{1 \pm \sqrt{5}\}$.

b. Apply the square root property.

$$(x+2) = \pm(2x+3)$$

Next, solve for x.

$$\begin{array}{rcr} x+2 = 2x+3 & \text{or} & x+2 = -(2x+3) \\ x-2x = 3-2 & & x+2 = -2x-3 \\ -x = 1 & & x+2x = -3-2 \\ x = -1 & & x = -\frac{5}{3} \end{array}$$

The solution set is $\left\{-1, -\frac{5}{3}\right\}$.

Completing The Square

To solve the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ by completing the square proceed as follows:

1. If $a \neq 1$, divide both sides of the equation by a.

2. Rearrange the equation so that the constant term is on the right side of the equal sign.

3. Square half the coefficient of x, and add this to both sides of the equation.

4. Factor the resulting trinomial and combine terms on the right side.

5. Apply the square root property to obtain the solution.

Example 14 **Solve by Using the Completing the Square**

Solve by completing the square method.

a. $x^2 + 6x - 16 = 0$

b. $2x^2 - 3x - 1 = 0$

Solution

a. $x^2 + 6x - 16 = 0$

$x^2 + 6x = 16$ • Isolate the constant term.

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 = 16 + \left(\frac{6}{2}\right)^2$$ • Add $\left(\frac{6}{2}\right)^2$ to both sides.

$x^2 + 6x + 9 = 16 + 9$

$(x+3)^2 = 25$ • Factor and simplify.

$x + 3 = \pm\sqrt{25}$ • Apply the square root property.

$x = -3 \pm 5$

The solution set is $\{2, -8\}$.

b. Divide both sides of the equation by the coefficient of x^2.

$$x^2 - \frac{3}{2}x - \frac{1}{2} = 0$$

Isolate the constant term.

$$x^2 - \frac{3}{2}x = \frac{1}{2}$$

Add the square of half the x coefficient to both sides.

$$x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 = \frac{1}{2} + \left(-\frac{3}{4}\right)^2$$

Factor and simplify.

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Apply the square root property.

$$x - \frac{3}{4} = \pm\sqrt{\frac{17}{16}}$$

Solve for x.

$$x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$$

$$x = \frac{3 \pm \sqrt{17}}{4}$$

The solution set is $\left\{ \frac{3+\sqrt{17}}{4}, \frac{3-\sqrt{17}}{4} \right\}$.

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 15 **Solve by Using the Quadratic Formula**

Solve using the quadratic formula.

a. $2x^2 - 3x - 1 = 0$

b. $-2 + 5x - 2x^2 = 0$

Solution

a. With coefficients $a = 2$, $b = -3$, $c = -1$, the quadratic formula yields

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{9+8}}{4}$$

$$x = \frac{3 \pm \sqrt{17}}{4}$$

The solution set is $\left\{ \frac{3+\sqrt{17}}{4}, \frac{3-\sqrt{17}}{4} \right\}$.

b. Write the equation in standard form.

$$-2x^2 + 5x - 2 = 0$$

The coefficients are $a = -2$, $b = 5$, and $c = -2$.

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(-2)(-2)}}{2(-2)}$$

$$x = \frac{-5 \pm \sqrt{25 - 16}}{-4}$$

$$x = \frac{-5 \pm \sqrt{9}}{-4}$$

$$x = \frac{-5 \pm 3}{-4}$$

The solution set is $\left\{ 2, \frac{1}{2} \right\}$.

The Discriminant

In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the expression $b^2 - 4ac$ is called the **discriminant.** When the coefficients $a, b,$ and c are real numbers, the sign of the discriminant determines whether there will be real roots or complex roots.

CLASSIFICATION OF ROOTS

The quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, has discriminant $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, then the quadratic equation has two distinct real roots.
- If $b^2 - 4ac = 0$, then the quadratic equation has a repeated root.
- If $b^2 - 4ac < 0$, then the quadratic equation has two distinct complex roots.

Example 16 **Use the Discriminant to Classify Roots**

Classify the roots of each equation.

a. $4x^2 - 3x - 1 = 0$
b. $x^2 - 4x + 4 = 0$
c. $3x^2 - 4x + 8 = 0$

Solution

a. The coefficients of $4x^2 - 3x - 1 = 0$ are $a = 4, b = -3$, and $c = -1$.

$$b^2 - 4ac = (-3)^2 - 4(4)(-1) = 9 + 16 = 25 > 0$$

Since the discriminant is positive, the quadratic equation has two distinct real solutions.

b. The coefficients of $x^2 - 4x + 4$ are $a = 1, b = -4$, and $c = 4$.

$$b^2 - 4ac = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

Since the discriminant is zero, the quadratic equation has a repeated root.

c. The coefficients of $3x^2 - 4x + 8 = 0$ are $a = 3, b = -4$, and $c = 8$.

$$b^2 - 4ac = (-4)^2 - 4(3)(8) = 16 - 96 = -80 < 0$$

Since the discriminant is negative, the quadratic equation has two distinct complex solutions.

Applications

In a right triangle the side opposite the 90° angle is called the **hypotenuse**, and the other two sides are just called **legs.** The **Pythagorean Theorem** states that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs. That is, in a right triangle shown in **Figure 2.1,** $c^2 = a^2 + b^2$. The Pythagorean Theorem is used to solve applications that involve right triangles.

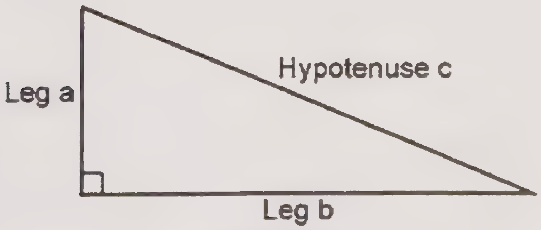


Figure 2.1

Example 17 **Solve a Geometric Application**

Suppose the length of a rectangular flower bed is twice its width, and the area is 800 square feet. Find the dimensions of the flower bed.

Solution

Let $l = 2w$

$A = lw$ • Area of a rectangle.

Since area is 800,

$lw = 800$

Substitute for l in the above equation to get

$$2w \cdot w = 800$$
$$2w^2 = 800$$
$$w^2 = 400$$
$$w = \pm 20$$

The negative solution is discarded, as the width of a rectangle can not be negative. Therefore, dimensions of the flower bed are: $w = 20$, and $l = 40$.

Example 18 **Solve a Geometric Application**

The length of a rectangular picture frame is 25 centimeters. Find the length of the diagonal of the frame, if the area is 425 cm^2.

Solution

Let $l =$ length of the frame

$w =$ width of the frame

$d =$ diagonal of the frame

Given that the length is 25 and the area is 425, we substitute these values into the area formula, $A = lw$, to get

$$425 = 25 \cdot w$$
$$w = 17$$

Now we use the Pythagorean Theorem to find the length of the diagonal.

$$d^2 = l^2 + w^2$$
$$d^2 = (25)^2 + (17)^2$$
$$d^2 = 625 + 289$$
$$d^2 = 914$$
$$d = \pm\sqrt{914}$$

The negative solution is discarded, as the diagonal of a rectangle can not be negative. Thus,

$$d = \sqrt{914} \approx 30.23$$

The diagonal of the rectangle is 30.23 cm.

2.4 OTHER TYPES OF EQUATIONS

Factorable - Type Equations

This method can be used to solve any equation that can be factored and is equal to zero. To find solutions use zero product property.

Example 19 **Solve an Equation that can be Factored**

Solve: $4x^3 = x$

Solution

$$\begin{aligned} 4x^3 &= x \\ 4x^3 - x &= 0 \\ x\left(4x^2-1\right) &= 0 \\ x = 0 \quad \text{or} \quad 4x^2 - 1 &= 0 \\ x^2 &= \frac{1}{4} \\ x &= \pm\frac{1}{2} \end{aligned}$$

The solution set is $\left\{0, \frac{1}{2}, -\frac{1}{2}\right\}$.

Radical Equations

To solve radical equations containing one radical

1. Isolate the radical expression on one side of the equation.
2. Raise both sides of the equation to the power that is equal to the index of the radical.
3. Simplify the equation.
4. Solve the equation obtained in step 3.
5. Check the proposed solutions in the *original equation.*

CAUTION: *Any solution that is not a solution of the original equation is called an* **extraneous solution.** *Extraneous solutions may be introduced when both sides of an equation is raised to an even power, both sides of an equation is raised to a rational power, or both sides of an equation is multiplied by a variable quantity. Thus, when any of these operations are performed we must check the proposed solution in the original equation.*

Example 20 **Solve an Equation Involving a Radical Equation**

Solve:

a. $\sqrt[3]{x-2} = 3$

b. $\sqrt{3x-2} + x = 4$

Solution

a. $\sqrt[3]{x-2} = 3$

$x - 2 = 27$ • Cube both sides.

$x = 29$

Check for $x = 29$: $\sqrt[3]{x-2} = 3$

$\sqrt[3]{\mathbf{29} - 2} = ?$

$$\sqrt[3]{27} = ?$$
$$3 = 3$$

Thus, $x = 29$ checks and the solution set is $\{29\}$.

b. The equation $\sqrt{3x-2}+x = 4$ can be solved by isolating the radical expression.

$$\sqrt{3x-2} = 4 - x$$

Next, square both sides as follows.

$$\left(\sqrt{3x-2}\right)^2 = (4-x)^2$$
$$3x - 2 = 16 - 8x + x^2$$

Collect and combine like terms.

$$x^2 - 11x + 18 = 0$$

Factor and solve the equation.

$$(x-2)(x-9) = 0$$
$$x = 2 \quad \text{or} \quad x = 9$$

Now, the proposed solutions must be checked in the original equation.

If $x = \mathbf{2}$

$$\sqrt{3x-2}+x = 4$$
$$\sqrt{3(\mathbf{2})-2}+\mathbf{2} = 4\ ?$$
$$\sqrt{6-2}+2 = 4\ ?$$
$$\sqrt{4}+2 = 4\ ?$$
$$2+2 = 4 \quad \textbf{True}$$

If $x = \mathbf{9}$

$$\sqrt{3x-2}+x = 4$$
$$\sqrt{3(\mathbf{9})-2}+\mathbf{9} = 4\ ?$$
$$\sqrt{27-2}+9 = 4\ ?$$
$$\sqrt{25}+9 = 4\ ?$$
$$14 \neq 4 \quad \textbf{False}$$

As this check shows, only 2 is a solution. Therefore, the solution set is $\{2\}$.

To solve radical equations with two or more square root expressions

1. Isolate the most complicated square root on one side of the equation.
2. Raise both sides to the power of 2.
3. Repeat steps 1&2 until all radicals are eliminated.
4. Simplify and solve the equation.
5. Check the solutions in the *original equation.*

Example 21 **Solve an Equation involving Two Radicals**

Solve: $\sqrt{2x+7} - \sqrt{x+3} = 1$

Solution

$$\sqrt{2x+7} - \sqrt{x+3} = 1$$
$$\sqrt{2x+7} = 1 + \sqrt{x+3}$$
$$\left(\sqrt{2x+7}\right)^2 = \left(1+\sqrt{x+3}\right)^2$$
$$2x + 7 = 1 + 2\sqrt{x+3} + x + 3$$
$$2x + 7 - 1 - x - 3 = 2\sqrt{x+3}$$
$$x + 3 = 2\sqrt{x+3}$$
$$(x+3)^2 = \left(2\sqrt{x+3}\right)^2$$
$$x^2 + 6x + 9 = 4(x+3)$$
$$x^2 + 6x + 9 - 4x - 12 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = -3 \quad \text{or} \quad x = 1$$

Check for $x = -3:\ \sqrt{2x+7} - \sqrt{x+3} = 1$

$$\sqrt{2(\mathbf{-3})+7} - \sqrt{\mathbf{-3}+3} = ?$$
$$\sqrt{-6+7} - 0 = 1$$

Check for $x = 1:\ \sqrt{2x+7} - \sqrt{x+3} = 1$

$$\sqrt{2(\mathbf{1})+7} - \sqrt{\mathbf{1}+3} = ?$$
$$\sqrt{2+7} - \sqrt{4} = ?$$
$$\sqrt{9} - 2 = 1$$

Both $x = -3$ and $x = 1$ check as solutions. The solution set is $\{-3, 1\}$.

Power Equations

Any equation of the form $x^{\frac{m}{n}} = b$, where m and n are integers, $n \neq 0$, and b is a real number, is called a **power equation.**

To solve power equations consider the following cases:

1. If $x^m = b$, $b > 0$ and m is a positive even integer, then there are two distinct real solutions, $x = \pm\sqrt[m]{b}$. For example, if $x^4 = 3$, then the two real solutions are $\pm\sqrt[4]{3}$.

2. If $x^m = b$, $b < 0$ and m is a positive even integer, then there are no real solutions. For example $x^2 = -3$ has no real solution.

3. If $x^m = b$, $b < 0$ and m is a positive odd integer, then there is one real solution, $x = \sqrt[m]{b}$. For example, if $x^3 = -5$, then the only real root is $\sqrt[3]{-5}$ or $-\sqrt[3]{5}$.

4. If $x^{\frac{m}{n}} = b$, b is a real number, then there is one solution, $x = b^{\frac{n}{m}}$. To solve $x^{\frac{m}{n}} = b$, raise both *sides* to the power of n, to obtain $x^m = b^n$. The equation $x^m = b^n$ may then be solved by taking the mth root of both sides.

Example 22 **Solve a Power Equation**

Solve: $x^{\frac{3}{2}} = 8$

Solution

First, raise both sides to the 2nd power. Then, solve for x.

$$x^{\frac{3}{2}} = 8$$

$$\left(x^{\frac{3}{2}}\right)^2 = (8)^2$$

$$x^3 = 64$$

$$x = 4$$

A check will verify that 4 is a solution.

Note: When raising both sides of an equation to a positive *even* integer power, it is essential to check all solutions.

Example 23 **Solve Power Equations Involving Rational Exponents**

Solve:

a. $2x^{\frac{3}{4}} - 16 = 0$

b. $x^{\frac{2}{3}} + \frac{2}{9} = \frac{1}{3}$

c. $\left(x^2 + x + 3\right)^{\frac{3}{2}} = 27$

Solution

The goal is to isolate the expression with rational exponent .

a. $2x^{\frac{3}{4}} - 16 = 0$

$$2x^{\frac{3}{4}} = 16$$

$$x^{\frac{3}{4}} = 8 \qquad \bullet \text{ Divide both sides by 2.}$$

$$\left(x^{\frac{3}{4}}\right)^4 = (8)^4 \qquad \bullet \text{ Raise both sides to the power of 4.}$$

$$x^3 = 4096$$
$$x = \sqrt[3]{4096} = 16$$

Check for $x = 16$:

$$2(\mathbf{16})^{\frac{3}{4}} - 16 = ?$$
$$2(8) - 16 = 0$$

The solution set is $\{16\}$.

b. $x^{\frac{2}{3}} + \frac{2}{9} = \frac{1}{3}$

$$x^{\frac{2}{3}} = -\frac{2}{9} + \frac{1}{3}$$

- Isolate the expression with rational exponents.

$$x^{\frac{2}{3}} = \frac{1}{9}$$

$$\left(x^{\frac{2}{3}}\right)^3 = \left(\frac{1}{9}\right)^3$$

- Raise both sides to the power of 3.

$$x^2 = \frac{1}{729}$$

$$x = \pm\frac{1}{\sqrt{729}} = \pm\frac{1}{27}$$

Since both sides of the equation are raised to a *positive odd integer power*, we do not need to check for extraneous solutions.

Therefore, the solution set is $\left\{\frac{1}{27}, -\frac{1}{27}\right\}$.

c. $(x^2 + x + 3)^{\frac{3}{2}} = 27$

$$\left[(x^2 + x + 3)^{\frac{3}{2}}\right]^2 = [27]^2$$
$$(x^2 + x + 3)^3 = 729$$
$$x^2 + x + 3 = \sqrt[3]{729}$$
$$x^2 + x + 3 = 9$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -3 \qquad\qquad x = 2$$

It can be shown that both proposed solutions are solutions of the original equation. The solution set is $\{-3, 2\}$. Since both sides of the equation are raised to a *positive even integer* power we had to check for extraneous solutions.

Quadratic - Type Equations

An equation is said to be of the **quadratic-type** if it can be written as

$$au^2 + bu + c = 0$$

where a, b, and c are real numbers, $a \neq 0$, and u is an algebraic expression.

The procedure to solve an equation quadratic in form is illustrated in the following example.

Example 24 **Solve an Equation that is Quadratic in Form**

Solve: $x^6 - 5x^3 + 6 = 0$

Solution

$x^6 - 5x^3 + 6 = 0$

$u^2 - 5u + 6 = 0$ • Make the substitution $u = x^3$.

$(u-3)(u-2) = 0$ • Factor.

$u - 3 = 0$ or $u - 2 = 0$

$u = 3$ or $u = 2$

$\downarrow$ $\downarrow$

$x^3 = 3$ $x^3 = 2$

$x = \sqrt[3]{3}$ $x = \sqrt[3]{2}$ • Take the cube root of both sides.

A check will verify that both $\sqrt[3]{3}$ and $\sqrt[3]{2}$ are solutions. Therefore, the solution set is $\{\sqrt[3]{3}, \sqrt[3]{2}\}$.

Note: Following is a table of equations that are quadratic in form. Each equation is followed by an appropriate substitution that will enable it to be written in quadratic form $au^2 + bu + c = 0$.

Original Equation	**Substitution**	**Quadratic Form**
$x^4 - 11x^2 + 24 = 0$	$u = x^2$	$u^2 - 11u + 24 = 0$
$x^6 + 2x^3 - 8 = 0$	$u = x^3$	$u^2 + 2u - 8 = 0$
$x^{\frac{1}{2}} - 9x^{\frac{1}{4}} + 18 = 0$	$u = x^{\frac{1}{4}}$	$u^2 - 9u^2 + 18 = 0$
$2x^{\frac{2}{3}} + x^{\frac{1}{3}} - 3 = 0$	$u = x^{\frac{1}{3}}$	$2u^2 + u - 3 = 0$
$12\left(\frac{1}{x}\right)^2 - 17\left(\frac{1}{x}\right) + 6 = 0$	$u = \frac{1}{x}$	$12u^2 - 17u + 6 = 0$

Example 25 **Solve an Equation That is Quadratic in Form**

Solve: $4x^{\frac{1}{2}} - 9x^{\frac{1}{4}} + 2 = 0$

Solution

$4x^{\frac{1}{2}} - 9x^{\frac{1}{4}} + 2 = 0$

$4u^2 - 9u + 2 = 0$ • Make the substitution $u = x^{\frac{1}{4}}$.

$(4u-1)(u-2) = 0$ • Factor

$4u - 1 = 0$ or $u - 2 = 0$

$u = \frac{1}{4}$ $u = 2$

$\downarrow$ $\downarrow$

$x^{\frac{1}{4}} = \frac{1}{4}$ $x^{\frac{1}{4}} = 2$ • Replace u with $x^{\frac{1}{4}}$.

$\left(x^{\frac{1}{4}}\right)^4 = \left(\frac{1}{4}\right)^4$ $\left(x^{\frac{1}{4}}\right)^4 = (2)^4$ • Raise both sides to the power of 4.

$x = \frac{1}{256}$ $x = 16$

Since both sides of the equation are raised to a *positive even integer* power, we must check for extraneous solutions. A check will verify that both $\frac{1}{256}$ and 16 are solutions. Therefore, the solution set is $\left\{\frac{1}{256}, 16\right\}$.

2.5 INEQUALITIES

Properties of Inequalities

ADDITION AND SUBTRACTION PROPERTY

If you add or subtract the same quantity on both sides of an inequality, the resulting inequalities are equivalent.

$a < b$, $a + c < b + c$ and $a - c < b - c$ are equivalent.

MULTIPLICATION AND DIVISION PROPERTY

If you Multiply or divide both sides of an inequality by the same positive number($c > 0$), the resulting inequalities are equivalent.

$a < b$, $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$ are equivalent.

If you Multiply or divide both sides of an inequality by the same negative number($c <$ the inequality sign reverses.

$a < b$, $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$ are equivalent.

Linear Inequalities

To solve linear inequalities proceed as follows:

1. Simplify each side of the inequality by eliminating fractions, removing parentheses, and combining like terms.
2. Use addition property to isolate the variable on the left side of the inequality.
3. Use the division property to write the inequality as $x < a$ or $x > a$.

A **solution** of an inequality is a value of the variable that makes the inequality a true statement. The **solution set** of an inequality is the set of all solutions. For example the solution set of the inequality $x < 3$ contains all numbers less than 3. In **set notation** the solution is $\{x \mid x < 3\}$, and in **interval notation** the solution is $(-\infty, 3)$.

Example 26 **Solve an Inequality**

Solve: $2x - 5 \leq 1 - 2(3x - 1)$

Solution

$2x - 5 \leq 1 - 2(3x - 1)$

$2x - 5 \leq 1 - 6x + 2$ • Use Distributive property.

$8x \leq 8$ • Add 6x and 5 to each side.

$x \leq 1$ • Divide each side by 8.

The solution set is $\{x \mid x \leq 1\}$, which in interval notation is $(-\infty, 1]$.

Example 27 **Solve an Application that Involves an Inequality**

A rental company charges \$12.00 to rent a chain saw, plus \$2.00 per hour. Mary can spend no more than \$30 to clear some logs from her front yard. What is the maximum amount of time she can keep the chain saw?

Solution

Let h be the number of hours she can rent the saw. She must pay \$12.00, plus \$$2h$, to rent the saw for h hours, and this amount must not be more than \$30.00.

$$\begin{aligned} 12 + 2h &\leq 30 \\ 2h &\leq 18 \\ h &\leq 9 \end{aligned}$$

Therefore, she can keep the chain saw no more than 9 hours.

Compound Inequalities

A **compound inequality** consists of two inequalities that are connected with the connective word *or* or *and*. The solution set of a compound inequality with the connective word *or* is the union of the solutions of the two inequalities. The solution set of a compound inequality with the connective word *and* is the intersection of the solutions of the two inequalities.

Example 28 **Solve Compound Inequalities**

Solve each compound inequality.

a. $2x + 1 > 3 \quad or \quad x + 5 < 4$

b. $2x + 5 > 5 \quad and \quad x - 4 > 0$

Solution

a. $2x + 1 > 3 \quad or \quad x + 5 < 4$

$x > 1 \qquad\qquad x < -1$

The solution set is $\{x \mid x > 1 \quad or \quad x < -1\} = \{x \mid x > 1\} \cup \{x \mid x < -1\}$

b. $2x + 5 > 5 \quad and \quad x - 4 > 0$

$x > 0 \qquad\qquad x > 4$

The solution is $\{x \mid x > 0\} \cap \{x \mid x > 4\} = \{x \mid x > 4\}$

Example 29 **Solve an Application Involving Compound Inequality**

A student has scores of 96, 85, 77 and 93 (on a scale of 100) on her four history tests. Her fifth grade should be in what range so that her average is more than 80, but less than 90?

Solution:

Let x be her score on the fifth test.

Since her average is less than 90 but more than 80,

$$80 < \frac{96 + 85 + 77 + 93 + x}{5} < 90$$

$$80 < \frac{351 + x}{5} < 90$$

$$400 < 351 + x < 450$$

$$49 < x < 99$$

Thus, she needs to make a grade between 49 and 99.

Other Inequalities

In this section we solve inequalities that contain polynomials of degree 2, as well as some that contain rational expressions. An inequality written with a polynomial on the left side and zero on the right side of the inequality is called a **polynomial inequality** in standard form. An inequality written with a rational expression on the left side and zero on the right side of the inequality is called a **rational inequality** in standard form.

2.5 INEQUALITIES

Polynomial and rational inequalities can be solved using the critical value method.

THE CRITICAL VALUE METHOD

1. Write the inequality in standard form.
2. Find the critical values.

   For rational inequalities, set the denominator and numerator equal to zero to find the real zeros of the rational expression. They are critical values of the inequality.

   For polynomial inequalities, set the polynomial equal to zero to find the real zeros of the polynomial expression. They are critical values of the inequality.

3. Place the critical values on the real number line and find the intervals into which these points divide the line.
4. In each interval, select a test number. Use these test numbers to determine the sign of the factors of the polynomial *or* numerator and denominator of the rational expression.

Example 30 Solve a Polynomial Inequality

Solve: $x^2 - x \leq 20$

Solution

First, write the inequality in standard form,

$x^2 - x - 20 \leq 0$

To find the critical values of the inequality set the expression equal to zero and solve for x.

$x^2 - x - 20 = 0$

$(x-5)(x+4) = 0$

$x - 5 = 0$ or $x + 4 = 0$

$x = 5$ or $x = -4$

The critical values, 5 and -4, divide a number line into the three regions as shown below.

	$-\infty < x \leq -4$	$-4 \leq x \leq 5$	$5 \leq x < \infty$
$x - 5$	$-$	$-$	$+$
$x + 4$	$-$	$+$	$+$
$(x-5)(x+4)$	$+$	$-$	$+$
Test numbers	-5	1	6

In the interval $-4 \leq x \leq 5$, $x^2 - x - 20 \leq 0$. Thus, the solution set is $\{x \mid -4 \leq x \leq 5\}$

Example 31 Solve a Rational Inequality

Solve: $\dfrac{2x-1}{x+1} \geq 1$

Solution

Begin by writing the rational inequality in standard form.

$$\frac{2x-1}{x+1} - 1 \geq 0$$

$$\frac{x-2}{x+1} \geq 0$$

To find the critical values of the inequality set the numerator and denominator equal to zero and solve for x.

Critical value of numerator is : $x = 2$

Critical value of denominator is : $x = -1$

The critical values, 2 and -1, divide a number line into the three regions as shown below.

	$-\infty < x < -1$	$-1 < x \leq 2$	$2 \leq x < \infty$
$x - 2$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$
$\dfrac{x-2}{x+1}$	$+$	$-$	$+$
Test numbers	-2	0	3

In the interval $2 \leq x < \infty$ and $-\infty < x < -1$, $\dfrac{x-2}{x+1} \geq 0$. Thus, the solution set is $\{x \mid 2 \leq x < \infty \textit{ or } -\infty < x < -1\}$.

2.6 ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Absolute Value Equations

Let X be an algebraic expression, and

If k is a positive real number, then the equation
$|X| = k$ *is equivalent to* $X = k$ *or* $X = -k$.
If k is a negative real number, then the equation
$|X| = k$ *has no solution.*
If k is zero, then the equation
$|X| = k$ *is equivalent to* $X = 0$.

Example 32 **Solve Absolute Value Equations**

Solve:
a. $|2x - 5| = 3$
b. $|x - 5| = -3$
c. $|x - 4| = 0$

Solution
a. $|2x - 5| = 3$
$2x - 5 = 3$ *or* $2x - 5 = -3$
$2x = 8$ $\quad 2x = 2$
$x = 4$ $\quad x = 1$
The solution set is $\{4, 1\}$.

b. $|x - 5| = -3$
Absolute value of a number is never negative, so there are no solutions to this equation. The solution set is $\emptyset$.

c. $|x - 4| = 0$
This absolute value equation is equivalent to $x - 4 = 0$. The solution set is $\{4\}$.

Absolute Value Inequalities

If X is an algebraic expression and k is a positive real number, then
$|X| < k$ *is equivalent to* $-k < X < k$.
$|X| > k$ *is equivalent to* $X < -k$ *or* $X > k$.

Remark: To solve absolute value inequalities make certain that the absolute value inequality is by itself on the left side and the positive constant is on the right side of the inequality. The rules hold if $<$ is replaced by $\leq$ and $>$ by $\geq$.

Example 33 **Solve an absolute Value Inequality with $>$**

Solve: $|2x - 3| > 5$

Solution
$|2x - 3| > 5$ is equivalent to $2x - 3 < -5$ *or* $2x - 3 > 5$. We will solve each of these inequalities.
$2x - 3 < --5$ *or* $2x - 3 > 5$
$2x < -2$ $\quad 2x > 8$
$x < -1$ $\quad x > 4$
The solution set is $\{x \mid x < -1 \textit{ or } x > 4\}$, which is $(-\infty, -1) \cup (4, \infty)$ in interval notation.

Example 34 **Solve an Absolute Value Inequality with $\leq$**

Solve: $\left|\frac{x-3}{4}\right| \leq 1$

Solution

$\left|\frac{x-3}{4}\right| \leq 1$

$-1 \leq \frac{x-3}{4} \leq 1$

$-4 \leq x-3 \leq 4$ • Multiply each side by 4.

$-1 \leq x \leq 7$ • Subtract 3 from all parts.

The solution set is $\{x \mid -1 \leq x \leq 7\}$, which is $[-1, 7]$ in interval notation.

If X is an algebraic expression and k is a negative real number, then

$|X| < k$ or $|X| \leq k$ *has no solution*

$|X| > k$ or $|X| \geq k$ *has infinite number of solutions.*

Example 35 **Solve Absolute Value Inequalities**

Solve:

a. $|2x+3| \leq -4$

b. $6\,|x-3| > -12$

Solution

a. $|2x+3| \leq -4$

Since $|2x+3|$ will always be greater than or equal to 0 for any real number x, this inequality can never be true. Thus, the solution is the empty set $\emptyset$.

b. $6\,|x-3| > -12$

$|x-3| > -2$ • Divide each side by 6.

Since $|x-3|$ will always be greater than or equal to 0 for any real number x, this inequality is true for all real numbers. Thus, the solution is the set of all real numbers.

If X is an algebraic expression and k is zero. We will show these cases by way of examples.

1. $|x-3| < 0$ has no solution because absolute value can never be negative.

2. $|x+8| \leq 0$ is equivalent to $x+8=0$. Therefore, the solution is $\{-8\}$.

3. $|2x-5| > 0$ the solution is every real number except $\frac{5}{2}$. Let us see why?

$$|2x-5| > 0 \text{ is equivalent to } \begin{array}{ccc} 2x-5<0 & \text{or} & 2x-5>0 \\ \downarrow & & \downarrow \\ x<\frac{5}{2} & & x>\frac{5}{2} \end{array}$$

Therefore, the solution set is $\left(-\infty, \frac{5}{2}\right), \left(\frac{5}{2}, \infty\right)$.

4. $|x-5| \geq 0$ the solution is the set of all real numbers, as any value of x will result in absolute value being greater than or equal to 0.

SOLVED PROBLEMS

Linear Equations

Solve each equation.

1. $7x = 2x - 10$

Solution

$7x = 2x - 10$

$7x-\mathbf{2x} = 2x - 10-\mathbf{2x}$ • Subtract 2x from both sides.

$5x = -10$

$\frac{5x}{\mathbf{5}} = -\frac{10}{\mathbf{5}}$ • Divide each side by 5.

$x = -2$

The solution set is $\{-2\}$.

2. $\frac{2}{3}x - \frac{x}{2} - 2 = \frac{1}{2}$

Solution

$\frac{2}{3}x - \frac{x}{2} - 2 = \frac{1}{2}$

$\mathbf{6}\left(\frac{2}{3}x - \frac{x}{2} - 2\right) = \mathbf{6}\left(\frac{1}{2}\right)$ • Multiply each side by 6, the LCD.

$4x - 3x - 12 = 3$

$x - 12 = 3$

$x - 12 + \mathbf{12} = 3 + \mathbf{12}$ • Add 12 to both sides.

$x = 15$

The solution set is $\{15\}$.

3. $(2x + 3)(x - 4) = 2(x^2 - 1)$

Solution

$(2x + 3)(x - 4) = 2(x^2 - 1)$

$2x^2 - 5x - 12 = 2x^2 - 2$ • Simplify each side.

$2x^2 - 5x - 12 - \mathbf{2x^2} = 2x^2 - 2 - \mathbf{2x^2}$ • Subtract $2x^2$ from both sides.

$-5x - 12 = -2$

$-5x - 12 + \mathbf{12} = -2 + \mathbf{12}$ • Add 12 to each side.

$-5x = 10$

$x = -2$ • Divide each side by -5.

The solution set is $\{-2\}$.

4. $\frac{-2}{x - 3} = \frac{3x}{2(x - 3)} - 1$

Solution

$\frac{-2}{x - 3} = \frac{3x}{2(x - 3)} - 1$

Begin by multiplying each side by $2(x - 3)$, the LCD, to eliminate the fractions.

$\mathbf{2(x-3)}\left(\frac{-2}{x - 3}\right) = \mathbf{2(x-3)}\left(\frac{3x}{2(x - 3)} - 1\right)$

$-4 = 3x - 2(x - 3)$

$-4 = 3x - 2x + 6$ • Simplify and combine terms.

$-4 - 6 = x$

$x = -10$

The solution set is $\{-10\}$.

5. $\dfrac{x}{x-5} + 2 = \dfrac{5}{x-5}$

Solution

$$\frac{x}{x-5} + 2 = \frac{5}{x-5}$$

To eliminate fractions, multiply each side by $x - 5$.

$$(\mathbf{x}-\mathbf{5})\left(\frac{x}{x-5}+2\right) = (\mathbf{x}-\mathbf{5})\left(\frac{5}{x-5}\right)$$

$x + 2(x-5) = 5$

$x + 2x - 10 = 5$

$3x = 15$

Divide each side by 3.

$$\frac{3x}{\mathbf{3}} = \frac{15}{\mathbf{3}}$$

$x = 5$

The apparent solution of the equation is 5, but substitution of it in the original equation results in denominator of zero. Thus, the solution set is $\{\}$.

6. $\dfrac{2}{x+3} - \dfrac{3}{x+2} = \dfrac{3}{x^2+5x+6}$

Solution

$$\frac{2}{x+3} - \frac{3}{x+2} = \frac{3}{x^2+5x+6}$$

Begin by factoring the denominator on the right side.

$$\frac{2}{x+3} - \frac{3}{x+2} = \frac{3}{(x+2)(x+3)}$$

$$(\mathbf{x}+\mathbf{3})(\mathbf{x}+\mathbf{2})\left[\frac{2}{x+3} - \frac{3}{x+2}\right] = (\mathbf{x}+\mathbf{3})(\mathbf{x}+\mathbf{2})\left[\frac{3}{(x+2)(x+3)}\right]$$

$2(x+2) - 3(x+3) = 3$

$2x + 4 - 3x - 9 = 3$

$-x - 5 = 3$

$-x = 8$ or $x = -8$

The solution set is $\{-8\}$.

Solve each literal equation for the given variable.

7. $V = \frac{1}{3}\pi r^2 h$; h

Solution

$V = \dfrac{1}{3}\pi r^2 h$

$\mathbf{3}(V) = \mathbf{3}\left(\dfrac{1}{3}\pi r^2 h\right)$ • Multiply each side by 3.

$3V = \pi r^2 h$

$h = \dfrac{3V}{\pi r^2}$ • Divide each side by πr^2.

8. $\dfrac{1}{f} = \dfrac{1}{a} + \dfrac{1}{b}$; a

Solution

$\dfrac{1}{f} = \dfrac{1}{a} + \dfrac{1}{b}$

To eliminate fractions, multiply each side by fab.

$\mathbf{fab}\left(\dfrac{1}{f}\right) = \mathbf{fab}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$

$ab = fb + fa$

Subtract fa from each side to group a terms on one side.

$ab - \mathbf{fa} = fb + fa - \mathbf{fa}$

$a(b-f) = fb$

$a = \dfrac{fb}{b-f}$ • Divide each side by $b - f$.

9. $y - y_1 = m(x - x_1)$; x_1

Solution

$y - y_1 = m(x - x_1)$

$y - y_1 = mx - mx_1$ • Apply the distributive property.

$y - y_1 - \mathbf{mx} = mx - mx_1 - \mathbf{mx}$ • Subtract mx from each side.

$y - y_1 - mx = -mx_1$

$\dfrac{y - y_1 - mx}{-\mathbf{m}} = \dfrac{-mx_1}{-\mathbf{m}}$ • Divide both sides by $-m$.

$\dfrac{y - y_1 - mx}{-m} = x_1$

$\left(\dfrac{-\mathbf{1}}{-\mathbf{1}}\right)\left(\dfrac{y - y_1 - mx}{-m}\right) = x_1$ • Eliminate the negative in denominator.

$x_1 = \dfrac{-y + y_1 + mx}{m}$

10. $F = \dfrac{Gm_1m_2}{d^2}$; m_2

Solution

$\mathbf{d^2}F = \mathbf{d^2}\dfrac{Gm_1m_2}{d^2}$

$d^2F = Gm_1m_2$

$m_2 = \dfrac{d^2F}{Gm_1}$

11. $A = p + prt$; t

Solution

$A = p + prt$

$A - \mathbf{p} = p + prt - \mathbf{p}$

$A - p = prt$

$t = \dfrac{A - p}{pr}$

Integer Problems

12. The sum of the squares of two consecutive positive integers is 145. Find the integers.

Solution

Let x and $x+1$ represent the two consecutive positive integers. Thus,

$$x^2 + (x+1)^2 = 145$$
$$x^2 + x^2 + 2x + 1 = 145$$
$$2x^2 + 2x - 144 = 0$$
$$x^2 + x - 72 = 0$$
$$(x-8)(x+9) = 0$$
$$x - 8 = 0 \quad \text{or} \quad x + 9 = 0$$
$$x = 8 \quad \text{or} \quad x = -9$$

Since -9 is a negative integer it is discarded. Given $x = 8$, $x + 1 = 9$. Therefore the two consecutive positive integers are 8 and 9.

13. Two-fifth of a number added to one-fourth of the number is 13. Find the number.

Solution

Let $x =$ the number.

Thus,

$$\frac{2}{5}x + \frac{1}{4}x = 13$$
$$20\left(\frac{2}{5}x + \frac{1}{4}x\right) = 20\,(13)$$
$$8x + 5x = 260$$
$$13x = 260$$
$$x = 20$$

The number is 20.

Investment Problems

14. Michelle has invested part of her money in mutual funds at an annual simple interest of 10%, and \$4,000 less than that amount in bonds at an annual simple interest rate of 6%. At the end of the year she earned \$5,600 in interest. How much was invested at each rate?

Solution

Let $x =$ amount invested at 10%

$x - 4{,}000 =$ amount invested at 6%

Using $I = prt$, with $t = 1$ year, gives

	principal ×	**rate** ×	**time** =	**interest**
Mutual fund	x	0.10	1	$0.10\,(x)$
Bond	$x - 4{,}000$	0.06	1	$0.06\,(x - 4{,}000)$

The sum of interest earned on the two accounts at the end of one year equals \$5,600. Therefore, the sum of the interest column must equal \$5,600.

$$0.10x + 0.06\,(x - 4{,}000) = 5{,}600$$
$$0.10x + 0.06x - 240 = 5{,}600$$
$$0.16x - 240 = 5{,}600$$
$$0.16x = 5{,}840$$
$$x = \$36{,}500 \quad @\ 10\%$$
$$x - 4{,}000 = \$32{,}500\ @\ 6\%.$$

SOLVED PROBLEMS

Work Problems

15. A technician and an assistant can repair a car in 5 hours. The technician can complete the repair in 9 hours. How long would it take the assistant to repair the car alone?

Solution

Let $t =$ the number of hours needed by the assistant to finish the task alone.

	rate ×	**time** =	**part of job completed**
Technician	$\frac{1}{9}$	5	$\frac{5}{9}$
Assistant	$\frac{1}{t}$	5	$\frac{5}{t}$

$$\begin{pmatrix}\textit{Part of job completed}\\ \textit{by the technician}\\ \textit{working with}\\ \textit{the assistant}\end{pmatrix} + \begin{pmatrix}\textit{Part of job completed}\\ \textit{by the assistant}\\ \textit{working with}\\ \textit{the technician}\end{pmatrix} = (\textit{One repaired car})$$

$$\frac{5}{9} + \frac{5}{t} = 1$$

$$\mathbf{9t}\left(\frac{5}{9} + \frac{5}{t}\right) = \mathbf{9t}\,(1)$$

$$5t + 45 = 9t$$

$$-4t = -45$$

$$t = \frac{45}{4} = 11\tfrac{1}{4}$$

Therefore, it takes the assistant $11\frac{1}{4}$ hours to repair the car alone.

Uniform Motion Problems

16. Suppose that two cyclists Ryan and Nicole live 120 miles apart, and at the same time they begin travelling towards each other. Ryan travels 3 miles per hour faster than Nicole. If they meet in 4 hours, find the speed of each cyclists.

Solution

Let $r = Ryan's\ speed$

$n = Nicole's\ speed$

Since Ryan travels 3 mph faster than Nicole, then

$r = n + 3$

	rate ×	**time** =	**distance**
Ryan	$n+3$	4	$4(n+3)$
Nicole	n	4	$4(n)$

The distance travelled by the cyclists is 120. Thus, the sum of the expressions in the distance column must equal 120.

$$4(n+3) + 4n = 120$$

$$4n + 12 + 4n = 120$$

$$8n = 120 - 12$$

$$8n = 108$$

$n = 13\frac{1}{2}\ mph$ Nicole's speed

$n + 3 = 16\frac{1}{2} mph$ Ryan's speed

17. Two cars start from cities 736 miles apart and travel toward each other. Car A travels at a constant speed of 55 miles per hour while Car B travels at a constant speed of 60 miles per hour. In how many hours will they meet?

Solution

Let $t =$ number of hours needed for cars to meet

	rate ×	**time** =	**distance**
Car A	55	t	$55t$
Car B	60	t	$60t$

Since the distance between the two cities is 736 miles, the sum of the expressions in the distance column must equal 736.

$$55t + 60t = 736$$
$$115t = 736$$
$$t = \frac{736}{115} = 6.40 \text{ hours}$$

Thus, the cars will meet after 6.40 hours..

18. A motorboat travels on a river whose current is running at 2.4 miles per hour. If the motorboat can travel 2 miles upstream in the same amount of time that it travels 5 miles downstream, find the speed of the boat.

Solution

Let $r =$ the average speed of boat in still water

Since the average speed of the current is 2.4 mph,

$r + 2.4 =$ the average speed of boat downstream

$r - 2.4 =$ the average speed of boat upstream

	distance ÷	**rate** =	**time**
Upstream	2	$r - 2.4$	$\frac{2}{r-2.4}$
Downstream	5	$r + 2.4$	$\frac{5}{r+2.4}$

Since the time to travel 5 miles *downstream* is the same as the time to travel 2 miles *upstream,* the expressions in the time column are equal.

$$\frac{5}{r+2.4} = \frac{2}{r-2.4}$$
$$5(r - 2.4) = 2(r + 2.4)$$
$$5r - 12 = 2r + 4.8$$
$$3r = 16.8$$
$$r = 5.6$$

The average speed of the boat is 5.6 $mph.$

Value Mixture Problems

19. How many pounds of a tea worth $5.60 must be mixed with 10 pounds of tea worth $2.00 to obtain a blend that sells for $3.10 per pound?

Solution

Let $x =$ pounds of $5.60 tea

	amount ×	**price** =	**value**
Type-1	x	5.60	$5.60x$
Type-2	10	2.00	$2(10)$
Mixture	$x + 10$	3.10	$3.10(x + 10)$

The sum of the values before mixing is equal to the value after mixing.

$5.60x + 2(10) = 3.10(x + 10)$
$5.6x + 20 = 3.1x + 31$
$2.5x = 11$
$x = 4.4$

The mixture should include 4.4 pounds of $5.60 tea.

20. How many ounces of pure water must be added to 12 ounces of a 40% Sulfuric acid to obtain a solution that is 25% Sulfuric acid?

Solution

Let $x =$ ounces of pure water

	amount ×	strength of acid =	amount of acid
Pure water	x	0	$0(x)$
40% solution	12	0.40	$12(0.40)$
25% solution	$12 + x$	0.25	$(12 + x)(0.25)$

$0(x) + 12(0.40) = (12 + x)(0.25)$
$4.8 = 3 + 0.25x$
$1.8 = 0.25x$
$x = 7.2$

The mixture must contain 7.2 ounces of pure water.

Coin Problems

21. A cashier box contains 200 nickels, dimes and quarters, and is worth $37. If there are 3 times as many dimes as nickels how many of each type of coin are there?

Solution

Let $x =$ number of nickels
$3x =$ number of dimes
$200 - 4x =$ number of quarters

	quantity ×	value =	total value
Nickles	x	0.05	$0.05(x)$
Dimes	$3x$	0.10	$0.10(3x)$
Quarters	$200 - 4x$	0.25	$0.25(200 - 4x)$

The sum of the total value of each denomination of coin equal the value of the collection.

$0.05x + 0.10(3x) + 0.25(200 - 4x) = 37$
$0.05x + 0.3x + 50 - x = 37$
$-0.65x = -13$
$x = 20$ nickels
$3x = 60$ dimes
$200 - 4x = 120$ quarters.

22. A collection of 112 coins which consist of nickels and dimes has a value of $10. Find the number of nickels in the collection.

Solution

Let $x =$ number of nickels

	quantity × value =		total value
Nickles	x	0.05	$0.05x$
Dimes	$112-x$	0.10	$0.10(112-x)$

The sum of the total value of each denomination of coin equals the value of the collection.

$0.05x+0.10(112-x)=10$
$0.05x+11.2-0.10x=10$
$-0.05x+11.2=10$
$-0.05x=-1.2$
$x=24$

The collection contains 24 nickels.

Business Problems

23. The price of a shoe rose 10% this year. If the shoe is priced at \$135 now, how much was its original price?

Solution

Let $x=$ the original price of the shoe

Price last year + increase = price this year

$x+0.10x=135$
$1.10x=135$
$x=122.73$

Thus, the original price of the shoe was about \$122.73.

24. The price of a math textbook is \$64 this year. This is 6% more than last year's price. What was last year's price?

Solution

Let $x=$ price of the book last year

Price this year = increase + price last year

$64=0.06x+x$
$64=1.06x$
$x=60.38$

The last year's price was about \$60.38.

Quadratic Equations

Solve each quadratic equation.

25. $x^2=5x$

Solution

$x^2=5x$
$x^2-5x=0$
$x(x-5)=0$
$x(x-5)=0$
$x=0 \quad \textit{or} \quad x-5=0$
$x=5$

The solution set is $\{0,5\}$.

26. $x^2-2x-3=0$

Solution

$x^2-2x-3=0$
$(x-3)(x+1)=0$
$x-3=0 \quad \textit{or} \quad x+1=0$
$x=3 \qquad x=-1$

The solution set is $\{-1, 3\}$.

27. $x(x-2) = 15$

Solution

$$x(x-2) = 15$$
$$x^2 - 2x - 15 = 0$$
$$(x-5)(x+3) = 0$$
$$x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 5 \qquad\qquad x = -3$$

The solution set is $\{5, -3\}$.

28. $(2x+5)^2 - 9 = 0$

Solution

$$(2x+5)^2 - 9 = 0$$
$$(2x+5)^2 = 9$$
$$2x + 5 = \pm 3$$
$$2x + 5 = 3 \quad \text{or} \quad 2x + 5 = -3$$
$$x = -1 \qquad\qquad x = -4$$

The solution set is $\{-4, -1\}$.

29. $x^2 = 4(x+3)$

Solution

$$x^2 = 4(x+3)$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$x - 6 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 6 \qquad\qquad x = -2$$

The solution set is $\{6, -2\}$.

Solve by completing the square.

30. $x^2 - 3x + 1 = 0$

Solution

$$x^2 - 3x + 1 = 0$$

$$x^2 - 3x = -1$$
- Isolate the constant term.

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = -1 + \left(-\frac{3}{2}\right)^2$$
- Add $\left(-\frac{3}{2}\right)^2$ to both sides.

$$x^2 - 3x + \frac{9}{4} = -1 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$$
- Factor and simplify.

$$x - \frac{3}{2} = \pm\sqrt{\frac{5}{4}}$$
- Apply the square root property.

$$x = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$
- Simplify.

The solution set is $\left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$

31. $2x^2 + 8x + 6 = 0$

Solution

$$2x^2 + 8x + 6 = 0$$

$$2x^2 + 8x = -6$$
- Isolate the constant term.

$$x^2 + 4x = -3$$
- Divide both sides by 2.

$$x^2 + 4x + 4 = -3 + 4$$
- Add $\left(\frac{4}{2}\right)^2$ to both sides.

$(x+2)^2 = 1$ • Factor and simplify.

$x+2 = \pm 1$ • Apply the square root property.

$x = -2 \pm 1$

The solution set is $\{-3, -1\}$.

Solve each equation by using the quadratic formula.

32. $x^2 + x + 2 = 0$

Solution

$x^2 + x + 2 = 0$

With coefficients $a = 1$, $b = 1$, and $c = 2$, the quadratic formula yields

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{1-8}}{2}$$
$$x = \frac{-1 \pm i\sqrt{7}}{2}$$

The solution set is $\left\{\frac{-1+i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}\right\}$.

33. $3x^2 + 6x = -1$

Solution

$3x^2 + 6x = -1$

Begin by writing the equation in standard form.

$3x^2 + 6x + 1 = 0$

With coefficients $a = 3$, $b = 6$, and $c = 1$, the quadratic formula yields

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(3)(1)}}{2(3)}$$
$$x = \frac{-6 \pm \sqrt{36-12}}{6}$$
$$x = \frac{-6 \pm \sqrt{24}}{6}$$
$$x = \frac{-6 \pm 2\sqrt{6}}{6}$$
$$x = -1 \pm \frac{\sqrt{6}}{3}$$

The solution set is $\left\{-1 + \frac{\sqrt{6}}{3}, -1 - \frac{\sqrt{6}}{3}\right\}$.

34. $x^2 + 6x + 9 = 0$

Solution

$x^2 + 6x + 9 = 0$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(9)}}{2}$$
$$x = \frac{-6 \pm \sqrt{36-36}}{2}$$
$$x = -\frac{6}{2} = -3$$

The solution set is $\{-3 \text{ repeated}\}$.

Classify the roots of each equation.

35. $x^2 - 5x + 7 = 0$

Solution

The coefficients of $x^2 - 5x + 7 = 0$ are $a = 1$, $b = -5$, and $c = 7$.

$$b^2 - 4ac = (-5)^2 - 4(1)(7) = 25 - 28 < 0$$

Since the discriminant is negative, the quadratic equation has two distinct complex solutions.

36. $3x^2 + 4x - 5 = 0$

Solution

The coefficients of $3x^2 + 4x - 5 = 0$ are $a = 3$, $b = 4$, and $c = -5$.

$$b^2 - 4ac = (4)^2 - 4(3)(-5) = 16 + 60 = 76 > 0$$

Since the discriminant is positive, the quadratic equation has two distinct real solutions.

37. The sum of the squares of three consecutive positive even integers is 116. Find the numbers.

Solution

Let $x, x + 2$, and $x + 4$ be the consecutive positive even integers.

$$x^2 + (x+2)^2 + (x+4)^2 = 116$$
$$x^2 + x^2 + 4x + 4 + x^2 + 8x + 16 = 116$$
$$3x^2 + 12x - 96 = 0$$
$$x^2 + 4x - 32 = 0$$
$$(x+8)(x-4) = 0$$
$$x + 8 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = -8 \qquad x = 4$$

The negative solution is discarded. The numbers are 4, 6, and 8.

38. Two cars left a junction at the same time, one heading south, and the other east. The car headed east had travelled 10 miles less than the car headed south. After one hour they were at a distance of 50 miles. Find the distance travelled by the car headed South.

Solution

Let x = the distance travelled by car headed south

$x - 10$ = the distance travelled by car headed east

50 = the distance between two cars after one hour

By the Pythagorean formula,

$$(50)^2 = x^2 + (x-10)^2$$
$$2500 = x^2 + x^2 - 20x + 100$$
$$0 = 2x^2 - 20x - 2400$$
$$0 = x^2 - 10x - 1200$$
$$0 = (x-40)(x+30)$$
$$x - 40 = 0 \quad \text{or} \quad x + 30 = 0$$
$$x = 40 \qquad x = -30$$

The negative solution is discarded, as the distance travelled can not be negative. Thus, the distance travelled by the car headed south is 40 $miles$.

39. An apartment complex swimming pool is rectangular in shape, 30 feet wide and 40 feet long. The management wants to build a walk around the pool. The walk is of uniform width and has an area of 456 square feet. How wide will the walk be?

Solution

Let x be the width of the walk as shown in Figure 2.2.

Area of rectangle − Area of pool = Area of walk

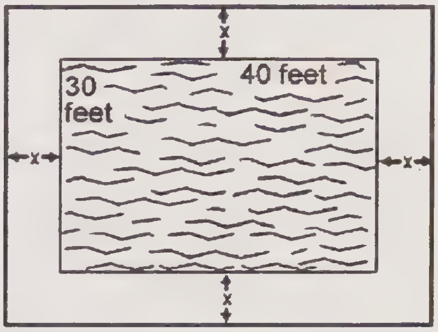


Figure 2.2

$$(30+2x)(40+2x)-1200=456$$
$$1200+60x+80x+4x^2-1200=456$$
$$4x^2+140x-456=0$$
$$x^2+35x-114=0$$
$$(x+38)(x-3)=0$$
$$x+38=0 \quad or \quad x-3=0$$
$$x=-38 \qquad\qquad x=3$$

The width can not be −38. Thus the width of the walk is 3 feet.

Other Types of Equations

Solve each equation, and check all solutions.

40. $x^3+2x^2-15x=0$

Solution

$$x^3+2x^2-15x=0$$
$$x\left(x^2+2x-15\right)=0$$
$$x(x-3)(x+5)=0$$

Apply the zero product property and solve.

$$x=0 \quad or \quad x-3=0 \quad or \quad x+5=0$$
$$x=3 \qquad\qquad x=-5$$

The solution set is $\{0,3,-5\}$.

41. $x^7-8x^4-x^3+8=0$

Solution

$$x^7-8x^4-x^3+8=0$$

Factor by grouping.

$$x^4\left(x^3-8\right)-\left(x^3-8\right)=0$$
$$\left(x^4-1\right)\left(x^3-8\right)=0$$
$$\left(x^2-1\right)\left(x^2+1\right)(x-2)\left(x^2+2x+4\right)=0$$
$$(x-1)(x+1)\left(x^2+1\right)(x-2)\left(x^2+2x+4\right)=0$$

Set each factor equal to zero and solve for x.

$$x-1=0 \qquad x=1$$
$$x+1=0 \qquad x=-1$$
$$x^2+1=0 \qquad x=\pm i$$
$$x-2=0 \qquad x=2$$
$$x^2+2x+4=0 \qquad x=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}$$

$$x=\frac{-2\pm 2\sqrt{-3}}{2}$$

$$x=\frac{-2\pm 2i\sqrt{3}}{2}$$

$$x=-1\pm i\sqrt{3}$$

The solutions are $\left\{\pm1,\pm i,2,-1\pm i\sqrt{3}\right\}$

Solve each radical equation, and check all solutions.

42. $\sqrt{3x+1}=\sqrt{x+7}$

Solution

$\sqrt{3x+1}=\sqrt{x+7}$
$3x+1=x+7$
$3x-x=7-1$
$2x=6$
$x=3$
Check for $x=3$: $\sqrt{3x+1}=\sqrt{x+7}$
$\sqrt{3\,(\mathbf{3})+1}=\sqrt{\mathbf{3}+7}$?
$\sqrt{9+1}=\sqrt{10}$?
$\sqrt{10}=\sqrt{10}$

Thus, $x=3$ checks and the solution set is $\{3\}$.

43. $\sqrt{3x+3+\sqrt{2x+3}}=1$

Solution

$\sqrt{3x+3+\sqrt{2x+3}}=1$
$3x+3+\sqrt{2x+3}=1$ • Square both sides.
$3x+3-1=-\sqrt{2x+3}$ • Isolate the radical expression.
$3x+2=-\sqrt{2x+3}$
$(3x+2)^2=\left(-\sqrt{2x+3}\right)^2$ • Square both sides.
$9x^2+12x+4=2x+3$
$9x^2+10x+1=0$ • Simplify.
$(9x+1)\,(x+1)=0$
$9x+1=0$ *or* $x+1=0$

$x=-\frac{1}{9}$ $x=-1$

Check for $x=-\frac{1}{9}$: $\sqrt{3x+3+\sqrt{2x+3}}=1$

$$\sqrt{3\left(-\frac{1}{9}\right)+3+\sqrt{2\left(-\frac{1}{9}\right)+3}}=?$$

$$\sqrt{-\frac{1}{3}+3+\sqrt{-\frac{2}{9}+3}}=?$$

$$\sqrt{\frac{8}{3}+\sqrt{\frac{25}{9}}}\neq 1$$

Thus, $x=-\frac{1}{9}$ does not check.

Check for $x=-1$: $\sqrt{3x+3+\sqrt{2x+3}}=1$

$\sqrt{3(-1)+3+\sqrt{2(-1)+3}}=?$

$\sqrt{-3+3+\sqrt{-2+3}}=?$

$\sqrt{0+\sqrt{1}}=1$

Thus, $x=-1$ checks as a solution, and the solution set is $\{-1\}$.

44. $\sqrt{5-x}+\sqrt{x+4}=3$

Solution

$\sqrt{5-x}+\sqrt{x+4}=3$

$\sqrt{5-x}=3-\sqrt{x+4}$ • Isolate one of the radicals.

$\left(\sqrt{5-x}\right)^2=\left(3-\sqrt{x+4}\right)^2$ • Square both sides.

$5-x=9-6\sqrt{x+4}+x+4$

$5-x-9-x-4=-6\sqrt{x+4}$ • Isolate the radical.

$-2x-8=-6\sqrt{x+4}$

$x+4=3\sqrt{x+4}$ • Divide through by -2.

$(x+4)^2=\left(3\sqrt{x+4}\right)^2$ • Square both sides.

$x^2+8x+16=9(x+4)$

$x^2+8x+16-9x-36=0$

$x^2-x-20=0$

$(x-5)(x+4)=0$ • *Factor.*

$x-5=0$ *or* $x+4=0$

$x=5$ $x=-4$

Check for $x=5$: $\sqrt{5-x}+\sqrt{x+4}=3$

$\sqrt{5-\mathbf{5}}+\sqrt{\mathbf{5}+\mathbf{4}}=?$

$\sqrt{0}+\sqrt{9}=3$

Check for $x=-4$: $\sqrt{5-x}+\sqrt{x+4}=3$

$\sqrt{5-(-4)}+\sqrt{-\mathbf{4}+\mathbf{4}}=?$

$\sqrt{9}+\sqrt{0}=3$

Since $x=5$ and $x=-4$ check as solutions, the solution set is $\{5,-4\}$.

Solve each equation, and check all solutions.

45. $x^{\frac{3}{2}}-4x^{\frac{1}{2}}=0$

Solution

$x^{\frac{3}{2}}=4x^{\frac{1}{2}}$

$x^3=64x$ • Raise both sides to the power of 2.

$x^3-64x=0$

$x\left(x^2-64\right)=0$

$x=0$ or $x=\pm4$

A check will verify that $x=-4$ is not the solution of the equation. therefore, the solution set is $\{0,4\}$.

46. $\left(x^2-2x\right)^{\frac{4}{3}}=16$

Solution

$\left(x^2-2x\right)^{\frac{4}{3}}=16$

$\left(x^2-2x\right)^4=16^3$ • Raise both sides to the power of 3.

$\left(x^2-2x\right)^4=4096$

$x^2-2x=8$ • Take the fourth root of both sides.

$x^2-2x-8=0$ • Subtract 8 from both sides.

$(x+2)(x-4)=0$ • Factor.

$x+2=0$ *or* $x-4=0$

$x = -2$ $\qquad\qquad$ $x = 4$

Note: Since both sides of the equation are raised to a *positive odd integer* power, we do not need to check for extraneous solutions. Therefore, the solution set is $\{-2, 4\}$.

47. $x^6 - 18x^3 + 81 = 0$

Solution

$$x^6 - 18x^3 + 81 = 0$$

Make the substitution of

$$u = x^3 \quad \text{and} \quad u^2 = x^6$$

to produce the quadratic equation

$$u^2 - 18u + 81 = 0$$
$$(u-9)(u-9) = 0$$
$$u - 9 = 0$$
$$u = 9$$

Substitute x^3 for u to get

$$x^3 = 9$$
$$x = \sqrt[3]{9} = (9)^{\frac{1}{3}}$$

Check for $x = 9^{\frac{1}{3}}$:

$$x^6 - 18x^3 + 81 = 0$$
$$\left(9^{\frac{1}{3}}\right)^6 - 18\left(9^{\frac{1}{3}}\right)^3 + 81 = 0\ ?$$
$$9^2 - 18(9) + 81 = 0\ ?$$
$$81 - 162 + 81 = 0$$

Thus, the solution set is $\{\sqrt[3]{9}\}$.

48. $\left(\dfrac{x}{x-1}\right)^2 - 2 = \dfrac{x}{x-1}$

Solution

$$\left(\frac{x}{x-1}\right)^2 - 2 = \frac{x}{x-1}$$

Write the equation in standard form.

$$\left(\frac{x}{x-1}\right)^2 - \frac{x}{x-1} - 2 = 0$$

Make the substitution of

$$u = \frac{x}{x-1} \quad \textit{and} \quad u^2 = \left(\frac{x}{x-1}\right)^2$$

to produce the quadratic equation

$$u^2 - u - 2 = 0$$
$$(u+1)(u-2) = 0$$
$$u + 1 = 0 \quad \textit{or} \quad u - 2 = 0$$
$$u = -1 \qquad\qquad u = 2$$

Substitute $\dfrac{x}{x-1}$ for u to get

$$\frac{x}{x-1} = -1 \quad \textit{or} \quad \frac{x}{x-1} = 2$$

$$-x + 1 = x \qquad\qquad 2x - 2 = x$$
$$-2x = -1 \qquad\qquad x = 2$$
$$x = \tfrac{1}{2}$$

Check for $x = 2$: $\left(\frac{x}{x-1}\right)^2 - 2 = \frac{x}{x-1}$

$$\left(\frac{2}{2-1}\right)^2 - 2 = \frac{2}{2-1} \quad ?$$

$$(2)^2 - 2 = 2 \quad ?$$

$$4 - 2 = 2 \quad ?$$

$$2 = 2$$

Check for $x = \frac{1}{2}$: $\left(\frac{x}{x-1}\right)^2 - 2 = \left(\frac{x}{x-1}\right)$

$$\left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right)^2 - 2 = \left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right) \quad ?$$

$$\left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right)^2 - 2 = \left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right) \quad ?$$

$$1 - 2 = -1$$

Thus, the solution set is $\left\{2, \frac{1}{2}\right\}$.

49. $(x^3 - 1)^2 + (x^3 - 1) - 20 = 0$

Solution

$$(x^3 - 1)^2 + (x^3 - 1) - 20 = 0$$

Make the substitution of

$$u = (x^3 - 1) \qquad \textit{and} \qquad u^2 = (x^3 - 1)^2$$

to produce the quadratic equation

$$u^2 + u - 20 = 0$$
$$(u + 5)(u - 4) = 0$$
$$u + 5 = 0 \qquad \textit{or} \qquad u - 4 = 0$$
$$u = -5 \qquad\qquad u = 4$$

Substitute $x^3 - 1$ for u to produce

$$x^3 - 1 = -5 \qquad \textit{or} \qquad x^3 - 1 = 4$$
$$x^3 = -4 \qquad\qquad x^3 = 5$$
$$x = \sqrt[3]{-4} \qquad\qquad x = \sqrt[3]{5}$$

Check for $x = \sqrt[3]{-4}$: $(x^3 - 1)^2 + (x^3 - 1) - 20 = 0$

$$\left[\left(\sqrt[3]{-4}\right)^3 - 1\right]^2 + \left[\left(\sqrt[3]{-4}\right)^3 - 1\right] - 20 =?$$

$$(-4 - 1)^2 + (-4 - 1) - 20 =?$$

$$25 - 5 - 20 = 0$$

Check for $x = \sqrt[3]{5}$: $(x^3 - 1)^2 + (x^3 - 1) - 20 = 0$

$$\left[\left(\sqrt[3]{5}\right)^3 - 1\right]^2 + \left[\left(\sqrt[3]{5}\right)^3 - 1\right] - 20 =?$$

$$(5 - 1)^2 + (5 - 1) - 20 =?$$

$$16 + 4 - 20 = 0$$

Thus, the solution set is $\left\{\sqrt[3]{4}, \sqrt[3]{5}\right\}$.

50. $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 3 = 0$

Solution

$$2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 3 = 0$$

Make the substitution of

$$u = x^{\frac{1}{3}} \qquad \textit{and} \qquad u^2 = x^{\frac{2}{3}}$$

to produce the quadratic equation

$$\begin{aligned} &2u^2 - u - 3 = 0 \\ &(2u-3)(u+1) = 0 \\ &2u - 3 = 0 \quad \text{or} \quad u + 1 = 0 \\ &u = \frac{3}{2} \qquad\qquad u = -1 \end{aligned}$$

Substitute $x^{\frac{1}{3}}$ for u to get

$$x^{\frac{1}{3}} = \frac{3}{2} \quad \text{or} \quad x^{\frac{1}{3}} = -1$$

$$x = \frac{27}{8} \qquad\qquad x = -1$$

Check for $x = \frac{27}{8}$:

$$\begin{aligned} 2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 3 &= 0 \\ 2\left(\frac{27}{8}\right)^{\frac{2}{3}} - \left(\frac{27}{8}\right)^{\frac{1}{3}} - 3 &=? \\ 2\left(\frac{9}{4}\right) - \frac{3}{2} - 3 &=? \\ \frac{9}{2} - \frac{3}{2} - 3 &= 0 \end{aligned}$$

Check for $x = -1$:

$$\begin{aligned} 2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 3 &= 0 \\ 2(-1)^{\frac{2}{3}} - (-1)^{\frac{1}{3}} - 3 &=? \\ 2 + 1 - 3 &= 0 \end{aligned}$$

Thus, the solution set is $\left\{\frac{27}{8}, -1\right\}$.

Inequalities

Solve each inequality.

51. $\frac{2x}{5} - \frac{3x-2}{15} \le -\frac{3}{10}$

Solution

$$\frac{2x}{5} - \frac{3x-2}{15} \le -\frac{3}{10}$$

$$30\left(\frac{2x}{5}\right) - 30\left(\frac{3x-2}{15}\right) \le 30\left(-\frac{3}{10}\right)$$ • Multiply each side by 30, the LCD.

$$12x - 6x + 4 \le -9$$

$$6x \le -13$$ • Subtract 4 from each side.

$$x \le -\frac{13}{6}$$ • Divide each side by 6.

The solution set is $\left\{x \mid x \le -\frac{13}{6}\right\}$, which in interval notation is $\left(-\infty, -\frac{13}{6}\right]$

52. $\frac{1}{3}(8x+3) - 7x \ge 2(4-x)$

Solution

$$\frac{1}{3}(8x+3) - 7x \ge 2(4-x)$$

$$8x + 3 - 21x \ge 6(4-x)$$ • Multiply both sides by 3.

$$-13x + 3 \ge 24 - 6x$$ • Use the distributive property and simplify.

$-13x \ge 21 - 6x$ • Subtract 3 from both sides.
$-7x \ge 21$ • Add 6x to both sides.
$x \le -3$ • Divide each side by -7.

The solution set is $\{x \mid x \le -3\}$, which is $(-\infty, -3]$ in interval notation.

53. $(x+4)(x-3) \ge (x+1)(x-2)$
Solution

$(x+4)(x-3) \ge (x+1)(x-2)$
$x^2 + x - 12 \ge x^2 - x - 2$ • Simplify each side.
$x - 12 \ge -x - 2$ • Subtract x^2 from each side.
$2x \ge 10$ • Add x and 12 to both sides.
$x \ge 5$ • Divide each side by 2.

The solution set is $\{x \mid x \ge 5\}$, which in interval notation is $[5, \infty)$.

54. $7x - 2 < 5$ *and* $5x - 10 > 5$
Solution

$7x - 2 < 5$ *and* $5x - 10 > 5$
$7x < 7$ $5x > 15$
$x < 1$ $x > 3$

The solution set is the set of all real numbers common to both solution sets. Clearly it is impossible to find such a set. Therefore, the solution set is $\{\}$.

55. $3x - 1 \ge 5$ *and* $4x - 5 \le 3$
Solution

$3x - 1 \ge 5$ *and* $4x - 5 \le 3$
$3x \ge 6$ $4x \le 8$
$x \ge 2$ $x \le 2$

The solution set is $\{2\}$.

56. $2x + 1 > 5$ *or* $4x - 3 < 13$
Solution

$2x + 1 > 5$ *or* $4x - 3 < 13$
$2x > 4$ $4x < 16$
$x > 2$ $x < 4$

The solution set is $\{x \mid x\textit{is all real numbers}\}$.

57. $\frac{1}{2} \le \frac{x-5}{4} + 2 \le \frac{5}{6}$
Solution
Begin by multiplying each part by 12, the LCD, to remove the denominators.

$$12\left(\frac{1}{2}\right) \le 12\left(\frac{x-5}{4} + 2\right) \le 12\left(\frac{5}{6}\right)$$

$6 \le 3(x-5) + 24 \le 10$ • Simplify.
$6 \le 3x - 15 + 24 \le 10$
$6 \le 3x + 9 \le 10$
$6 - 9 \le 3x + 9 - 9 \le 10 - 9$ • Subtract 9 from each part.
$-3 \le 3x \le 1$ • Simplify.

$-1 \le x \le \frac{1}{3}$ • Divide each part by 3.

The solution set is $\left\{x \mid -1 \le x \le \frac{1}{3}\right\}$, which in interval notation is $\left[-1, \frac{1}{3}\right]$.

58. $\frac{2x+5}{10} < \frac{x-5}{2} < \frac{2x}{5}$

Solution

$$\frac{2x+5}{10} < \frac{x-5}{2} < \frac{2x}{5}$$

$10\left(\frac{2x+5}{10}\right) < 10\left(\frac{x-5}{2}\right) < 10\left(\frac{2x}{5}\right)$ • Multiply each part by 10.

$2x+5 < 5(x-5) < 2(2x)$

$2x+5 < 5x-25 < 4x$

Recall that the inequality

$2x+5 < 5x-25 < 4x$

is equivalent to the two inequalities

$2x+5 < 5x-25$ *and* $5x-25 < 4x$

Solve each inequality separately.

The first inequality yields,

$2x+5 < 5x-25$

$2x+5-\mathbf{5x} < 5x-25-\mathbf{5x}$ • Subtract 5x from each side.

$-3x+5 < -25$

$-3x+5-\mathbf{5} < -\mathbf{25}-5$ • Subtract 5 from each side.

$-3x < -30$

$x > 10$ • Divide each side by -3.

The second inequality is solved as,

$5x-25 < 4x$

$5x-25+\mathbf{25} < 4x+25$ • Add 25 to both sides.

$5x < 4x+25$

$5x-\mathbf{4x} < 4x+25-\mathbf{4x}$ • Subtract 4x from each side.

$x < 25$

The solution set of the original problem is

$x > 10$ *and* $x < 25$

This solution set may be written as, $\{x \mid 10 < x < 25\}$, which in interval notation can be written as $(10, 25)$.

59. The perimeter of a rectangle is less than 33 cm, but greater than 21cm. The length is twice as long as the width. Find the range of the possible values for the length.

Solution

Since the perimeter of the rectangle ranges from 33 to 21,

$21 < perimeter < 33$

Let l equal the length of the rectangle and w the width.

The length is twice as long as the width, therefore

$l = 2w$

Also perimeter of the rectangle is given by

$p = 2l + 2w$

Substitute $\frac{l}{2}$ for w in the perimeter formula.

$$p = 2l + 2\left(\frac{l}{2}\right) = 2l + l = 3l$$

Thus,

$21 < 3l < 33$

$7 < l < 11$

The length is between 11 cm. and 7 cm.

60. John can rent a car during a weekend from company A for \$10.00 plus \$0.09, for each mile driven. Company B charges \$16 plus \$0.03 for each mile driven. Determine the range of mileage, that is cheaper to rent from company A?

Solution

Let $x =$ number of miles driven by the car.

The cost of renting from each company is:

Company A	$10 + 0.09x$
Company B	$16 + 0.03x$

If it is cheaper to rent from company A then,

$$\begin{aligned} 10 + 0.09x &< 16 + 0.03x \\ 0.09x - 0.03x &< 16 - 10 \\ 0.06x &< 6 \\ x &< 100 \end{aligned}$$

If John plans to drive less than 100 miles, then it is cheaper to rent from company A.

Use the critical value method to solve each inequality.

61. $\dfrac{x^2 - x - 12}{x - 2} \le 0$

Solution

$$\frac{x^2 - x - 12}{x - 2} \le 0$$

$$\frac{(x+3)(x-4)}{x-2} \le 0$$

Critical values of numerator: $x + 3 = 0, \quad x = -3$

$x - 4 = 0, \quad x = 4$

Critical values of denominator: $x - 2 = 0, \quad x = 2$

	$-\infty < x \le -3$	$-3 \le x < 2$	$2 < x \le 4$	$4 \le x < \infty$
$x + 3$	$-$	$+$	$+$	$+$
$x - 4$	$-$	$-$	$-$	$+$
$x - 2$	$-$	$-$	$+$	$+$
$\dfrac{(x+3)(x-4)}{x-2}$	$-$	$+$	$-$	$+$
Test numbers	-4	1	3	5

In the intervals $-\infty < x \le -3$ and $2 < x \le 4$ the expression $\dfrac{x^2 - x - 12}{x - 2}$ is zero or negative (≤ 0). Thus, the solution set is $\{x \mid -\infty < x \le -3 \text{ or } 2 < x \le 4\}$, which in interval notation is $(-\infty, -3] \cup (2, 4]$.

62. $x^2 \ge -8x - 12$

Solution

$$x^2 \ge -8x - 12$$

Write the inequality in standard form.

$$x^2 + 8x + 12 \ge 0$$

To find the critical values, set the expression equal to zero and solve for x.

$x^2 + 8x + 12 = 0$
$(x+2)(x+6) = 0$
$x + 2 = 0$ *or* $x + 6 = 0$
$x = -2$ $x = -6$

	$-\infty < x \leq -6$	$-6 \leq x \leq -2$	$-2 \leq x < \infty$
$x+2$	$-$	$-$	$+$
$x+6$	$-$	$+$	$+$
$(x+2)(x+6)$	$+$	$-$	$+$
Test numbers	-7	-3	-1

In the intervals $-\infty < x \leq -6$ and $-2 \leq x < \infty$, the expression $x^2 + 8x + 12$ is positive or zero (≥ 0). Thus, the solution set is $\{x \mid -\infty < x \leq -6 \text{ or } -2 \leq x < \infty\}$, which is $(-\infty, -6] \cup [-2, \infty)$ in interval notation.

Absolute Value Equations

Solve each equation.

63. $2|2x-5| + 3 = 7$

Solution

$2|2x-5| + 3 = 7$
$2|2x-5| + 3 - \mathbf{3} = 7 - \mathbf{3}$ • Subtract 3 from both sides.
$2|2x-5| = 4$
$|2x-5| = 2$ • Divide each side by 2.

$2x - 5 = 2$ *or* $2x - 5 = -2$
$2x = 7$ $2x = 3$

$x = \frac{7}{2}$ $x = \frac{3}{2}$

The solution set is $\left\{\frac{7}{2}, \frac{3}{2}\right\}$.

64. $|5x-2| = -2$

Solution

Absolute value of a number is always positive. Thus, the solution set is the empty set.

65. $|3x-2| = 0$

Solution

$|3x-2| = 0$
$3x - 2 = 0$

$x = \frac{2}{3}$

The solution set is $\left\{\frac{2}{3}\right\}$.

Absolute Value Inequalities

Solve each inequality. Write your answer in solution set.

66. $2|2x+3| > 6$

Solution

$2|2x+3| > 6$

$|2x+3|>3$ • Divide each side by 2.

$$\begin{array}{rcr} 2x+3<-3 & \textit{or} & 2x+3>3 \\ 2x<-6 & & 2x>0 \\ x<-3 & & x>0 \end{array}$$

The solution set is $\{x \mid x<-3 \textit{ or } x>0\}$, which is $(-\infty,-3)\cup(0,\infty)$ in interval notation.

67. $|2x+3|-2\le 7$

Solution

$|2x+3|-2\le 7$

$|2x+3|\le 9$ • Add 2 to each part.

$-9\le 2x+3\le 9$

$-9-\mathbf{3}\le 2x+3-\mathbf{3}\le 9-\mathbf{3}$ • Subtract 3 from each part.

$-12\le 2x\le 6$

$-6\le x\le 3$ • Divide each part by 2.

The solution is $\{x \mid -6\le x\le 3\}$, which is $[-6,3]$ in interval notation.

68. $|3x-5|+7\ge 6$

Solution

$|3x-5|+7\ge 6$

$|3x-5|\ge -1$

The solution set is $\{x \mid x \text{ is all real numbers}\}$.

69. $|x-5|\le -4$

Solution

$|x-5|\le -4$

Absolute value of a number must always be nonnegative. Thus, the solution set is $\{\}$.

70. $|x-5|\le 0$

Solution

Recall that "$\le$" means "less than or equal to." The absolute value of any expression will never be less than zero, but it may equal zero. Thus, to solve $|x-5|\le 0$ we solve $x-5=0$.

$x-5=0$

$x=5$

The solution set is $\{5\}$.

PRACTICE TEST

Solve.

1. $3(x-2)=6(2-x)+9$

2. $5+\frac{x}{3}=\frac{x+1}{5}$

3. $\frac{x}{x-5}+2=\frac{5}{x-5}$

Solve for the indicated variable.

4. $\frac{1}{f}=\frac{1}{a}+\frac{1}{b};\quad a$

5. $y-y_1=m(x-x_1);\quad x_1$

6. Use the discriminant to classify the roots of the equation $5x^2-13x+8=0$

Solve.

7. $(2x+5)^2-9=0$

8. $2x^{\frac{2}{3}}-x^{\frac{1}{3}}-3=0$

9. $\sqrt{2-2x}-3=x$

Solve any TWO of the following problems.

10. A technician and an assistant can repair a car in 5 hours. The technician can complete the repair in 9 hours. How long would it take the assistant to repair the car alone?

11. Rob invested \$35,000 into two accounts. One investment earned 6% and the other 8%. The amount of interest earned for one year was the same for both accounts. How much has he invested in each account?

12. How many ounces of pure water must be added to 12 ounces of a 40% sulfuric acid to obtain a solution that is 25% sulfuric acid?

Solve each inequality.
13. $x^2 \geq -2x + 15$

14. $|x + 2| - 4 = 5$

15. $|2 - 3x| \leq 5$

16. $|8x - 4| < -3$

17. $2x + 1 > 5$ *or* $4x - 3 < 13$

CHAPTER THREE

Functions and Graphs Part I

3.1 Rectangular Coordinate System
3.2 Relations and Functions
3.3 More about Functions
3.4 Graphing Techniques

3.1 RECTANGULAR COORDINATE SYSTEM

The rectangular coordinate plane consists of two real number lines. The horizontal line is usually called the $x - axis$ ($absissa$), and the vertical line the $y - axis$ ($ordinate$), and the point of intersection the $origin$. The axes divides the plane into four regions called **quadrants**. The quadrants are numbered in a counterclockwise order as shown in **Figure 3.1.**

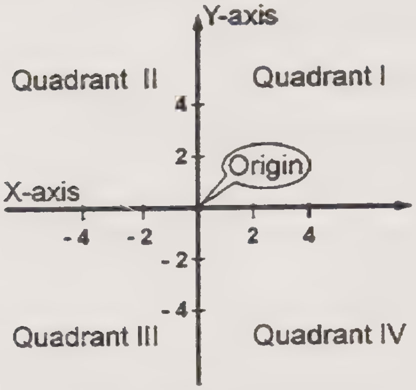


Figure 3.1

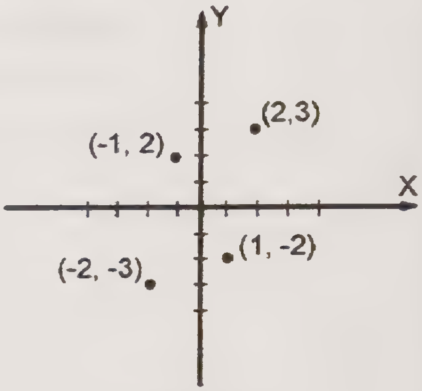


Figure 3.2

Any point A is plotted in the $xy - plane$ by using an **ordered pair** (x, y) of real numbers. The coordinates of point A is called an ordered pair, because the order is important. In **Figure 3.2** we have plotted points $(2,3)$, $(-1,2)$, $(-2,-3)$ and $(1,-2)$.

Note: *The points on the x-axis are denoted as (x, 0) and on the y-axis as (0, y).*

The Distance Formula

The distance d between the two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1 **Find the Distance Between Two Points**

Find the distance between two points $(-2,-3)$ and $(-5,1)$.
Solution
To find the distance d between the two points $P_1 = (-2,-3)$ and $P_2 = (-5,1)$, apply the distance formula. In using the formula, let $x_1 = -2$, $y_1 = -3$, $x_2 = -5$, and $y_2 = 1$.

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[-5 - (-2)]^2 + [1 - (-3)]^2} \\ &= \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5 \end{aligned}$$

The Midpoint Formula

The midpoint (x, y) of the line segment from (x_1, y_1) to (x_2, y_2) is

$$(x,y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Example 2 **Find the Midpoint of a Line Segment**

Find the midpoint of the line segment with endpoints $(0,7)$ and $(2,-3)$.

Solution

Let $(x_1,y_1) = (0,7)$ and $(x_2,y_2) = (2,-3)$. Use the midpoint formula to find the coordinates of the midpoint.

$$(x,y) = \left(\frac{0+2}{2}, \frac{7+(-3)}{2}\right) = (1,2)$$

Intercepts

The coordinate of the point where a graph intersects the $x-axis$ is called the *x-intercept.* The coordinate of the point where a graph crosses the $y-axis$ is called the *y-axis.*

Example 3 **Find Intercepts**

Find the $x-$and $y-$intercepts of the graph of each equation.

a. $y = x^2 - 9$

b. $x = y^2 + 2$

Solution

a. To find the $x-intercept$, let $y = 0$ and solve for x.

$$0 = x^2 - 9$$
$$9 = x^2$$
$$x = \pm 3$$

To find the $y-intercept$, let $x = 0$ and solve for y.

$$y = -9$$

Thus, the $x-intercepts$ are $(3,0)$ and $(-3,0)$. The $y-intercept$ is $(0,-9)$.

b. To find the $x-intercept$, let $y = 0$ and solve for x.

$$x = 2$$

To find the $y-intercept$, let $x = 0$ and solve for y.

$$0 = y^2 + 2$$
$$y^2 = -2$$
$$y = \pm\sqrt{-2}$$

Thus, the $x-intercept$ is $(2,0)$. There is no $y-intercept$.

Graph of a Linear Equation

The graph of a linear equation can be found using at least two points. The two points that are useful for sketching the graph of a linear equation are called *intercepts.*

Example 4 **Graph a Linear Equation**

Graph: $4x + y = 2$

Solution

To graph this equation first find the $x-$ and $y-intercepts$.

To find the $x-intercept$, let $y = 0$.

$$4x + 0 = 2$$
$$x = \tfrac{1}{2}$$

To find the $y - intercept$, let $x = 0$.

$$4(0) + y = 2$$
$$y = 2$$

Now, plot the two points $\left(\frac{1}{2}, 0\right)$ and $(0, 2)$ then connect them with a straight line.

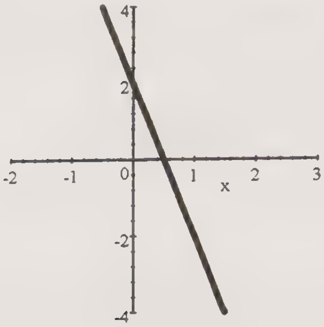


Figure 3.3

Circles

A circle is the set of points in a plane, which are at a fixed distance r from a given point. The distance is called the **radius,** and the given point is the **center** of the circle.

STANDARD FORM OF THE EQUATION OF A CIRCLE

Standard form of the equation of a circle with center at the origin and radius r is

$$x^2 + y^2 = r^2$$

Standard form of the equation of a circle with center at (h, k) and radius r is

$$(x - h)^2 + (y - k)^2$$

Example 5 **Find the Center and Radius of a Circle**

Find the center and radius of the circle $x^2 + y^2 = 9$.
Solution
The given equation is the equation of a circle with center at the origin and radius 3.

Example 6 **Write the Standard Form of the Equation of a Circle**

Find the standard form of the equation of a circle with radius 4 and center $(-2, 3)$.
Solution
We can find its equation by using the standard form of the equation of a circle.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (\mathbf{-2})]^2 + [y - \mathbf{3}]^2 = \mathbf{4}^2$$
$$(x + 2)^2 + (y - 3)^2 = 16$$

GENERAL FORM OF THE EQUATION OF A CIRCLE

General form of the equation of a circle is given by

$$x^2 + y^2 + ax + by + c = 0$$

Example 7 **Find the Center and Radius of a Circle**

Determine the center and radius of the circle that is given by

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

Solution

Step 1 Regroup and move constant to the right side.

$$\left(x^2 - 4x\right) + \left(y^2 - 6y\right) = -9$$

Step 2 Complete the square for x by adding $\left(-\frac{4}{2}\right)^2 = 4$ to each side of the equation.

$$\left(x^2 - 4x + \mathbf{4}\right) + \left(y^2 - 6y\right) = -9 + \mathbf{4}$$

Step 3 Complete the square for y by adding $\left(-\frac{6}{2}\right)^2 = 9$ to each side of the equation.

$$\left(x^2 - 4x + \mathbf{4}\right) + \left(y^2 - 6y + \mathbf{9}\right) = -9 + \mathbf{4} + \mathbf{9}$$

Step 4 Factor the perfect square trinomials.

$$(x-2)^2 + (y-3)^2 = 4$$

Thus, the center of this circle is at $(2, 3)$ and the radius 2.

Remark: To graph equation of a circle given in general form, it is best to write the equation in the standard form. This is achieved by using the method of completing the squares on both the x and y terms.

3.2 RELATIONS AND FUNCTIONS

In many situations in mathematics, science, and business it is useful to describe one quantity in terms of another. For example, the area of a circle is related to its radius, the demand for a product is related to its price.

The relationship between the area of a circle and its radius can be written as a set of ordered pairs in which the first number represents the radius, and the second number represents the area. In this example, the area of the circle depends on the radius. For this reason, area is called the *dependent variable,* and radius the *independent variable.*

Relations

A **relation** is any set of ordered pair. The **domain** of a relation is the set of all first coordinates of the ordered pairs (*values of x*). The **range** of the relation is the set of all second coordinates of the ordered pairs (*values of y*).

Functions

A function is a relation in which no ordered pairs have the same $x-$ *value.*

Example 8 **Test for Functions Represented by Graphs**

Which of the following graphs are graphs of functions?

a. **b.**

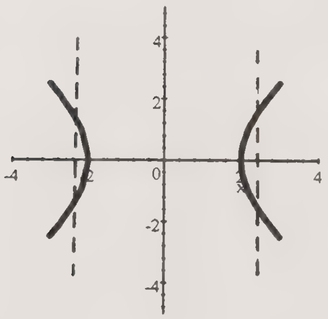


Figure 3.4

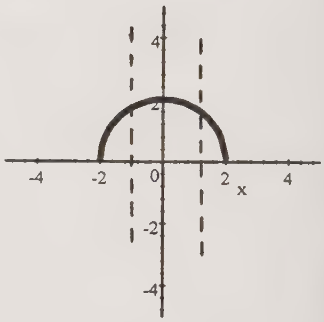


Figure 3.5

Solution

To determine if a graph is a graph of a function, use **vertical line test.** If a vertical line can be drawn so that it intersects the graph at most once, then the graph is the graph of a function.

a. This graph is not the graph of a function, as every vertical line intersects the graph in more than one point.

b. This graph is the graph of a function, because every vertical line intersects the graph in at most one point.

Example 9 **Test For Functions Represented by Equations**

Which of the equations define(s) y as a function of x?

a. $x = y^2 + 4$ **b.** $x^2 + y = 2$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields the following:

$$x = y^2 + 4$$
$$-y^2 = -x + 4$$
$$y^2 = x - 4$$
$$y = \pm\sqrt{x - 4}$$

The ± indicates that to a given value of x corresponds two values of y. Thus, this equation does not define y as a function of x.

b. Solving for y yields the following:

$$x^2 + y = 2$$
$$y = -x^2 + 2$$

To each value of x corresponds one value of y. Thus, y is a function of x.

Domain of a Function

Domain of a function defined by a *relation* is the set of all first coordinates in the set of ordered pairs.

Example 10 **Determine the Domain of a Function**

Find the domain of $H = \{(-3,0), (1,5), (2,6), (7,8)\}$

Solution

The domain of H consists of all first coordinates in the set of the ordered pairs.

$$\text{Domain} = \{-3, 1, 2, 7\}$$

Domain of a function defined by an *equation* is the set of all real numbers that can be substituted for x in the equation except:

- Those values of x that make the value of the function not real.
- Those values of x that make the denominator of the function zero.

Example 11 **Determine the Domain of a Function**

Find the domain of each function.

a. $f(x) = \sqrt{x - 5}$ **b.** $g(x) = \dfrac{2}{x+1}$

Solution

a. This function is defined only for $x-values$ for which $x - 5 \geq 0$, or when $x \geq 5$. Therefore,

$$D = \{x \mid x \geq 5\}$$

b. To find the domain of $g(x)$, set the denominator equal to zero and solve for x.

$$x + 1 = 0$$
$$x = -1$$

The domain of g is the set of all real numbers, except -1.

$$D = \{x \mid x \neq -1\}$$

Functional Notation

Functional notation is another way of finding y when y is a function of x or y is the dependent variable. Instead of writing $y = x^2 + 3x + 2$, we write $f(x) = x^2 + 3x + 2$. This is called **functional notation.** In this equation, x is the **independent variable** and y is the **dependent variable.** The symbol $f(x)$ is read as "f of x." The letters f, F, g, G, h, H are commonly used to represent functions.

Example 12 **Evaluate Functions**

Let $f(x) = x^3 - 4$, evaluate

a. $f(-1)$ **b.** $f(2)$

Solution

a. $\mathbf{f(-1) = (-1)^3 - 4 = -5}$

b. $\mathbf{f(2) = (2)^3 - 4 = 4}$

3.3 MORE ABOUT FUNCTIONS

INCREASING AND DECREASING FUNCTIONS

A function f is **increasing** on an interval if, for any choice of x_1, and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function f is **decreasing** on an interval if, for any choice of x_1, and x_2 in the interval $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Symmetry

SYMMETRY WITH RESPECT TO AN AXIS AND THE ORIGIN

A graph is **symmetric with respect to the x-axis** if whenever (x, y) is on the graph, then $(x, -y)$ is also on the graph. A graph is **symmetric with respect to the y-axis** if whenever (x, y) is on the graph, then $(-x, y)$ is also on the graph. A graph is **symmetric with respect to the origin** if whenever (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

TESTS FOR SYMMETRY

Given the equation of the graph, the following tests must be performed to determine types of symmetry.

The graph of an equation is symmetric with respect to

- the **x-axis** if replacement of y by $-y$ yields an equivalent equation.
- the **y-axis** if replacement of x by $-x$ yields an equivalent equation.
- the **origin** if replacement of x by $-x$ and y by $-y$ yields an equivalent equation.

Example 13 **Test Graph for Symmetries**

Determine whether the graph of $y = 3x^2 - 4$ is symmetric with respect to the $x - axis$, $y - axis$, or the *origin*.

Solution

a. Test for symmetry with respect to the **x-axis.**

Replace y with $-y$.

$-y = 3x^2 - 4$

This is not equivalent to the original equation. Therefore, the graph of $y = 3x^2 - 4$ is *not symmetric* with respect to the $x - axis$.

b. Test for symmetry with respect to the **y-axis.**

Replace x with $-x$.

$y = 3(-x)^2 - 4$

$y = 3x^2 - 4$

This yields an equivalent equation. Thus, the graph of $y = 3x^2 - 4$ is *symmetric* with respect to the $y - axis$.

c. Test for symmetry with respect to the **origin.**

Replace x with $-x$ and y with $-y$.

$-y = 3(-x)^2 - 4$

$-y = 3x^2 - 4$

This does not yield an equivalent equation, therefore the graph of $y = 3x^2 - 4$ is *not symmetric* with respect to the origin.

Even and Odd Functions

A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

The following properties are results of the tests for symmetry.

- The graph of an even function is symmetric with respect to the $y-axis$.
- The graph of an odd function is symmetric with respect to the $origin$.

Example 14 **Identify Even or Odd Functions**

Determine whether each function is odd, even or neither.
a. $f(x) = |x| + 2$ **b.** $f(x) = x^3$ **c.** $f(x) = 2x^5 + 1$

Solution

Replace x with $-x$ and simplify.
a. $f(-x) = |-x| + 2 = |x| + 2 = f(x)$
Since $f(-x) = f(x)$, we say $f(x)$ is an even function.

b. $f(-x) = (-x)^3 = -x^3 = -f(x)$
This function is an odd function since $f(-x) = -f(x)$.

c. $f(-x) = 2(-x)^5 + 1 = -2x^5 + 1$
This function is neither odd nor even since $f(-x) \neq f(x)$ or $f(-x) \neq -f(x)$.

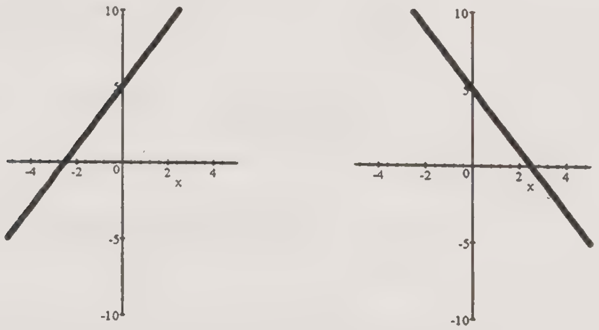
IMPORTANT FUNCTIONS AND THEIR GRAPHS

The graph of a function is the same as the graph of the equation $y = f(x)$. The graph of a function is the set of all points (x, y) such that x is in the domain of f.

Linear Function

A function of the form $f(x) = mx + b$, where m and b are real numbers, is a **linear function**. The graph of this function is a nonvertical straight line with slope m and $y-intercept$ b.

A linear function is increasing if $m > 0$, and decreasing if $m < 0$.
Domain: {all real numbers}



$m > 0$ $m < 0$

Figure 3.6

Constant Function

A function of the form $f(x) = b$, where b is a real number, is a **constant function**. The graph of this function is a horizontal line with slope zero and $y - intercept$ b. Below is the graph of $f(x) = 3$.

Domain: {all real numbers}
Range: $\{y \,/\, y = 3\}$

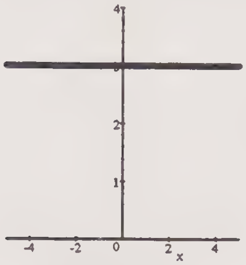


Figure 3.7

Identity Function

A function of the form $f(x) = x$, is an **identity function.** The graph of this function is a line whose slope is $m = 1$, and whose $y-intercept$ is zero. The identity function is an odd function, and the graph of it bisects quadrants I and III.

Domain: {all real numbers}
Range: {all real numbers}

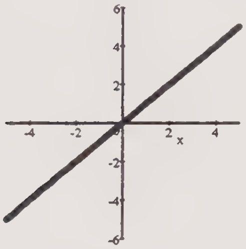


Figure 3.8

Square Function

A function of the form $f(x) = x^2$, is a **square function.** The graph of this function is a parabola, whose vertex is at $(0, 0)$. The square function is an even function.

Domain: {all real numbers}
Range: {all nonegative real numbers}

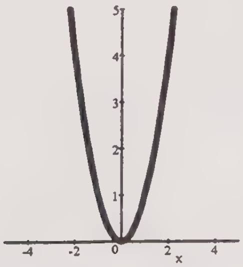


Figure 3.9

Cube Function

A function of the form $f(x) = x^3$, is called a **cube function.** The intercept of the graph is at $(0, 0)$. The cube function is an odd function.

Domain: {all real numbers}
Range: {all real numbers}

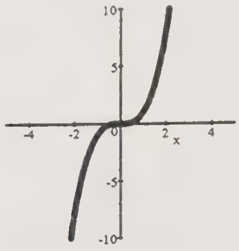


Figure 3.10

Square Root Function

A function of the form $f(x) = \sqrt{x}$, is called a **square root function.** The square root function is neither odd nor even.

Domain: {all nonnegative real numbers}
Range: {all nonnegative real numbers}

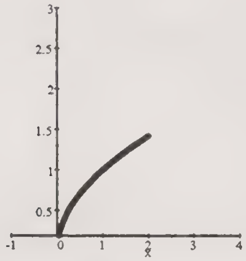


Figure 3.11

Reciprocal Function

A function of the form $f(x) = \dfrac{1}{x}$, is called a **reciprocal function.** The reciprocal function is an odd function and has no intercepts.

Domain: {all real numbers}
Range: { all real numbers}

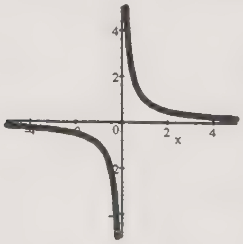


Figure 3.12

Absolute Value Function

A function of the form $f(x) = |x|$, is called an **absolute value function.** The absolute value function is an even function and the intercept of the graph is at $(0,0)$.

Domain: {all real numbers}
Range: {all nonnegative real numbers}

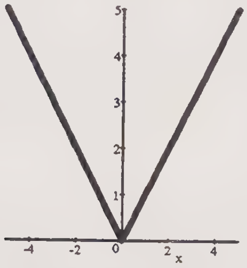


Figure 3.13

3.4 GRAPHING TECHNIQUES

TRANSLATION OF GRAPHS

Vertical Translation

If f is a function and c is a positive constant, then

$y = f(x) + c$ is the graph of $y = f(x)$ shifted up vertically c units.

$y = f(x) - c$ is the graph of $y = f(x)$ shifted down vertically c units.

Horizontal Translation

If f is a function and c is a positive constant, then

$y = f(x + c)$ is the graph of $y = f(x)$ shifted left horizontally c units.

$y = f(x - c)$ is the graph of $y = f(x)$ shifted right horizontally c units.

Example 15 **Graph Using Translation**

Graph using graphing techniques.

a. $f(x) = |x| + 2$ **b.** $f(x) = |x - 4|$

Solution

a. The graph of $f(x) = |x| + 2$ is the graph of $g(x) = |x|$ shifted up 2 units.

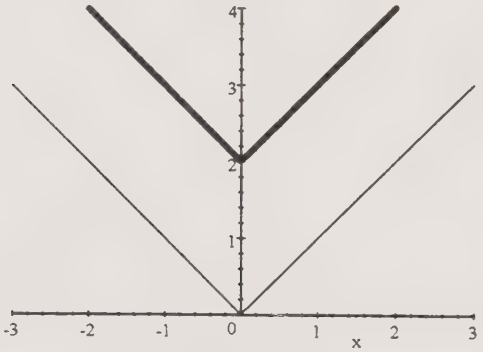


Figure 3.14

b. The graph of $f(x) = |x - 4|$ is the graph of $g(x) = |x|$ shifted 4 units to the right.

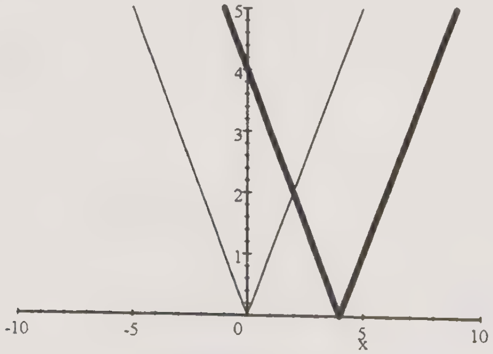


Figure 3.15

SHRINKING AND STRETCHING OF GRAPHS

Shrinking Principle

If f is a function and c is a positive constant with $0 < c < 1$, then $y = c\,f(x)$ is the graph of $y = f(x)$ shrinking by a factor of c toward the $x - axis$.

Stretching Principle

If f is a function and c is a positive constant with $c > 1$, then $y = c\,f(x)$ is the graph of $y = f(x)$ stretching by a factor of c toward the $y - axis$.

Example 16 **Graph Using Shrinking and Sketching Principles**

Graph each function using graphing techniques.

a. $f(x) = 2x^2$ **b.** $f(x) = \frac{1}{3}x^2$

Solution

a. The graph of $f(x) = 2x^2$ is the graph of $g(x) = x^2$, stretched by a factor of 2.

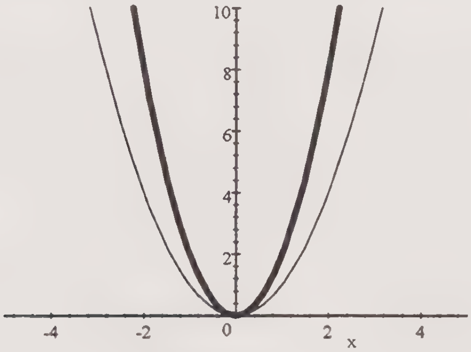


Figure 3.16

b. The graph of $f(x) = \frac{1}{3}x^2$ is the graph of $g(x) = x^2$, shrunk by a factor of $\frac{1}{3}$.

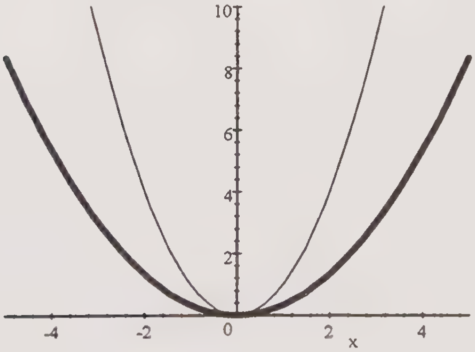


Figure 3.17

REFLECTION OF GRAPHS

Reflection Principle

If f is a function, then

$y = -f(x)$ is the graph of $y = f(x)$ reflected about the $x - axis.$

$y = f(-x)$ is the graph of $y = f(x)$ reflected about the $y - axis.$

Example 17 **Graph Using Reflections**

Graph each function using graphing techniques.

a. $f(x) = -\sqrt{x}$ **b.** $f(x) = \sqrt{-x}$

Solution

a. The graph of $f(x) = -\sqrt{x}$ is the graph of $g(x) = \sqrt{x}$ reflected about the $x - axis.$

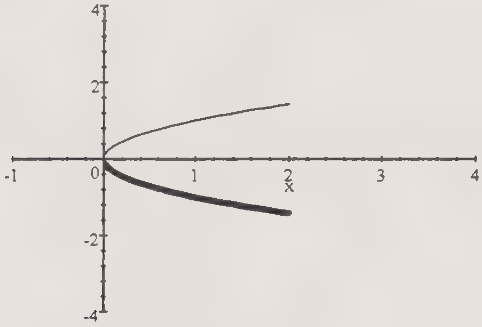


Figure 3.18

b. The graph of $f(x) = \sqrt{-x}$ is the graph of $g(x) = \sqrt{x}$ reflected about the $y - axis.$

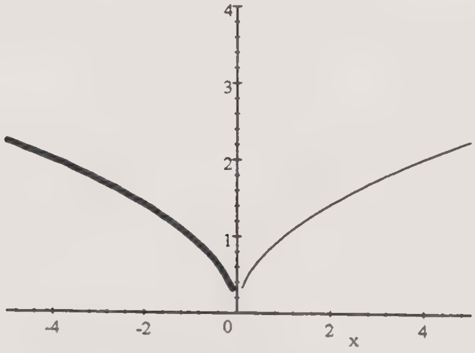


Figure 3.19

SOLVED PROBLEMS

Rectangular Coordinate System

Find the distance between the points whose coordinates are given.

1. $(-\sqrt{2}, \sqrt{3})$, $(2, 0)$

Solution

Let $P_1 = (-\sqrt{2}, \sqrt{3})$, $P_2 = (2, 0)$. Then, $x_1 = -\sqrt{2}$, $y_1 = \sqrt{3}$, $x_2 = 2$, and $y_2 = 0$.

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[2 - (-\sqrt{2})]^2 + [0 - \sqrt{3}]^2} \\ &= \sqrt{(2 + \sqrt{2})^2 + (-\sqrt{3})^2} \\ &= \sqrt{4 + 2 + 4\sqrt{2} + 3} \\ &= \sqrt{9 + 4\sqrt{2}} \approx 3.83 \end{aligned}$$

2. Find all the points with $x - coordinate$ of 3 whose distance from the point $(3, -4)$ is 5.

Solution

Let $P_1 = (3, y_1)$, $P_2 = (3, -4)$ and $d = 5$.

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$5 = \sqrt{(3 - 3)^2 + (-4 - y_1)^2}$$

$$25 = (4 + y_1)^2$$

$$\pm 5 = 4 + y_1$$ • Take the square root of each side

$$5 = 4 + y_1$$

or

$$-5 = 4 + y_1$$

$$y_1 = 1 \quad \text{or} \quad y_1 = -9$$

The points are $(3, 1)$ and $(3, -9)$.

3. Find all the points with $y - coordinate$ of 3 whose distance from the point $(2, 3)$ is 6.

Solution

Let $P_1 = (x_1, 3)$, $P_2 = (2, 3)$ and $d = 6$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$6 = \sqrt{(2 - x_1)^2 + (3 - 3)^2}$$

$$6 = \sqrt{(2 - x_1)^2}$$

$$36 = (2 - x_1)^2$$

$$\pm 6 = 2 - x_1$$ • Take the square root of each side.

$$6 = 2 - x_1$$

$-6 = 2 - x_1$

$x_1 = -4$ or $x_1 = 8$

The points are $(8,3)$ and $(-4,3)$.

4. Find the midpoint of the line segment joining the points $P_1\,(-2,-4)$ and $P_2\,(3,-5)$.

Solution

Let $(x_1,y_1) = (-2,-4)$ and $(x_2,y_2) = (3,-5)$.

$$(x,y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{-2+3}{2}, \frac{-4-5}{2}\right)$$

$$= \left(\frac{1}{2}, -\frac{9}{2}\right)$$

The midpoint is $\left(\frac{1}{2}, -\frac{9}{2}\right)$.

5. A line segment has a midpoint at $(2,-1)$ and an endpoint at $(4,-3)$. Find the coordinates of the other endpoint of the line segment.

Solution

Use the midpoint formula

$$(x,y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

The formula for the $x-$coordinate of the midpoint is $\frac{x_1+x_2}{2}$. Here the $x-$coordinate of the midpoint is 2, and $x_1 = 4$.

$$2 = \frac{4+x_2}{2}$$

$4 = 4 + x_2$

$0 = x_2$

The formula for the $y-$coordinate of the midpoint is $\frac{y_1+y_2}{2}$. Here the $y-$coordinate of the midpoint is -1, and $y_1 = -3$.

$$-1 = \frac{-3+y_2}{2}$$

$-2 = -3 + y_2$

$+1 = y_2$

Therefore, the coordinates of the other endpoint is $(0,1)$.

6. Find the length of the side of the triangle joining the vertices $(1,2)$ and $(3,-2)$.

Solution

Let $P_1 = (x_1,y_1) = (1,2)$ and $P_2 = (x_2,y_2) = (3,-2)$.

$$d(P_1,P_2) = \sqrt{(1-3)^2 + [2-(-2)]^2}$$

$$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

The length of the side of the triangle is $2\sqrt{5}$.

7. Show that the triangle whose vertices are $A = (3,2)$, $B = (-1,-1)$, and $C = (-4,-5)$, is an isosceles triangle (an isosceles triangle is one in which two

sides are of equal length). Then find the perimeter of the triangle.

Solution

Use the distance formula to find the length of each side of the triangle.

$$d(A,C) = \sqrt{[3-(-4)]^2 + [2-(-5)]^2} = \sqrt{49+49} = 7\sqrt{2}$$

$$d(A,B) = \sqrt{[3-(-1)]^2 + [2-(-1)]^2} = \sqrt{16+9} = 5$$

$$d(B,C) = \sqrt{[-1-(-4)]^2 + [-1-(-5)]^2} = \sqrt{9+16} = 5$$

It follows that the triangle ABC is an isosceles triangle.

$$Perimeter = 5+5+7\sqrt{2} = 10+7\sqrt{2}$$

8. Suppose two vertices of an equilateral triangle (an equilateral triangle is one in which all three sides are of equal length) are given as $(0,1)$, and $(0,3)$. Find the third vertex such that the triangle is located in the first quadrant.

Solution

Let $A=(0,1), B=(0,3), C=(x,y)$.

$$d(A,B) = \sqrt{(0-0)^2 + (1-3)^2} = \sqrt{4} = 2$$

Since all three sides of the triangle are equal,

$$d(A,B) = d(A,C) = d(B,C) = 2$$

The distance between A and C, written as $d(A,C)$ is

$$d(A,C) = \sqrt{(0-x)^2 + (1-y)^2} = 2$$

Square each side of the equation $\sqrt{x^2 + (1-y)^2} = 2$.

$$x^2 + (1-y)^2 = 4 \qquad (1)$$

The distance between B and C, written as $d(B,C)$ is

$$d(B,C) = \sqrt{(0-x)^2 + (3-y)^2} = 2$$

Square each side of the equation $\sqrt{x^2 + (3-y)^2} = 2$.

$$x^2 + (3-y)^2 = 4 \qquad (2)$$

Equate equations (1) and (2) to solve for y.

$$\begin{aligned} x^2 + (1-y)^2 &= x^2 + (3-y)^2 \\ (1-y)^2 &= (3-y)^2 \\ 1-2y+y^2 &= 9-6y+y^2 \\ 4y &= 8 \\ y &= 2 \end{aligned}$$

To find the $x-coordinate$ of the third vertex, substitute $y=2$ into equation (1).

$$x^2 + 1 = 4$$
$$x^2 = 3$$

$x = \pm\sqrt{3}$

Since the triangle is to be located in the first quadrant, $x = -\sqrt{3}$ is discarded. Thus, the third vertex is $\left(\sqrt{3}, 2\right)$.

9. Show that the triangle whose vertices are $A = (2, 4)$, $B = (2, -2)$, and $C = (-3, -2)$ is a right triangle. Then find area of the right triangle.

Solution

Plot the points and construct the triangle. Then use the distance formula.

$$d(A, B) = \sqrt{(2-2)^2 + [4-(-2)]^2} = \sqrt{36} = 6$$

$$d(A, C) = \sqrt{[2-(-3)]^2 + [4-(-2)]^2} = \sqrt{25+36} = \sqrt{61}$$

$$d(B, C) = \sqrt{[2-(-3)]^2 + [-2-(-2)]^2} = \sqrt{25} = 5$$

It seems that the hypotenuse is side AC (the longer side), the side opposite the right angle. Now we will check to see whether

$$[d(A, C)]^2 = [d(A, B)]^2 + [d(B, C)]^2$$

$$\left(\sqrt{61}\right)^2 = 6^2 + 5^2$$

$$61 = 36 + 25$$

So, by converse of the Pythagorean theorem the triangle is a right triangle.

$$Area = \tfrac{1}{2}\,(base)\,(height)$$

$$Area = \tfrac{1}{2}\,[d(B, C)]\,[d(A, B)]$$

$$= \tfrac{1}{2}\,(5)\,(6)$$

$$Area = 15 \ \ square\ units.$$

Graph of an Equation

Sketch the graph of each equation.

10. $y - 2x = 1$

Solution

Graph $y = 2x + 1$, using the intercepts.

The $x-intercept$ is found by letting $y = 0$.

$$0 - 2x = 1$$

$$-2x = 1$$

$$x = -\frac{1}{2}$$

The $y-intercept$ is found by letting $x = 0$.

$$y - 2(0) = 1$$

$$y = 1$$

Plotting the ordered pairs $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$ gives the graph in **Figure 3.20.**

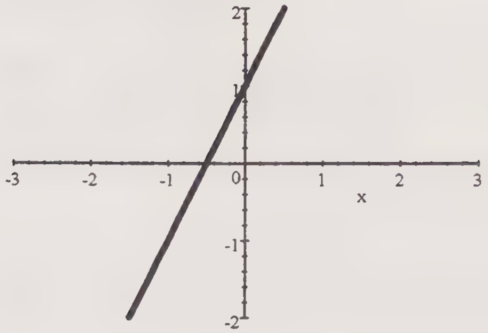


Figure 3.20

11. $3y - 6 = 0$
Solution

$$3y - 6 = 0$$
$$3y = 6$$
$$y = 2$$

Since y always equals 2, the value of y can never be zero. This means that the graph has no $x - intercept$. The only way a straight line has no $x - intercept$ is when it is parallel to the $x - axis$, as shown in **Figure 3.21.**

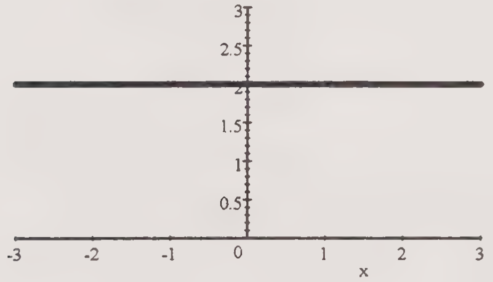


Figure 3.21

The graph of the equation $3y - 6 = 0$ is a horizontal line with slope, zero, and $y - intercept$, 2.

Intercepts

Find the x and y intercepts of the graph of each equation.

12. $x^3 = y - 27$
Solution

To find the $x - intercept$, let $y = 0$ and solve for x.

$$x^3 = -27$$
$$x = \sqrt[3]{-27}$$
$$x = -3$$

The $x - intercept$ is $(-3, 0)$.

To find the $y - intercept$, let $x = 0$ and solve for y.

$$0 = y - 27$$
$$y = 27$$

The $y-intercept$ is $(0,27)$.

13. $x = \dfrac{y^2-4}{y}$

Solution

To find the $x-intercept$, let $y=0$. But in here y cannot be zero. Therefore, there is no $x-intercept$.

To find the $y-intercept$, let $x=0$ and solve for y.

$$0 = \frac{y^2-4}{y}$$

$$\begin{aligned}0 &= y^2-4\\ 4 &= y^2\\ y &= \pm 2\end{aligned}$$

The $y-intercepts$ are $(0,2)$ and $(0,-2)$.

14. $|x-y| = 3$

Solution

To find the $x-intercept$, let $y=0$.

$$\begin{aligned}|x-0| &= 3\\ |x| &= 3\\ x &= \pm 3\end{aligned}$$

The $x-intercepts$ are $(3,0)$ and $(-3,0)$.

To find the $y-intercept$, let $x=0$.

$$\begin{aligned}|0-y| &= 3\\ |y| &= 3\\ y &= \pm 3\end{aligned}$$

The $y-intercepts$ are $(0,3)$ and $(0,-3)$.

15. $y = |x|-2$

Solution

To find the $x-intercept$, let $y=0$

$$\begin{aligned}0 &= |x|-2\\ 2 &= |x|\\ x &= \pm 2\end{aligned}$$

The $x-intercepts$ are $(2,0)$ and $(-2,0)$.

To find the $y-intercept$, let $x=0$.

$$y=-2$$

The $y-intercept$ is $(0,-2)$.

Circle

16. Determine the equation of a circle in standard form with center $(2,-2)$ and radius 4.

Solution

Use the equation of a circle in standard form with $h=2$, $k=-2$, and $r=4$.

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-2)^2+[y-(-2)]^2=4^2$$

The equation of the circle in standard form is

$$(x-2)^2+(y+2)^2=16$$

Determine the center and radius of the circle with the given equation.

17. $(x+2)^2+y^2=9$

Solution

The equation $(x+2)^2+y^2=9$ can be written as

$$[x-(-2)]^2+(y-0)^2=3^2$$

which appears to be the equation of a circle in standard form. This circle is centered at $(-2,0)$ with radius 3.

18. $x^2+y^2+8x+7=0$

Solution

Step 1: Regroup and move constant to the right side.

$$\left(x^2+8x\right)+y^2=-7$$

Step 2: Complete the square for x, by adding $\left(\frac{8}{2}\right)^2=16$ *to each side of the equation.*

$$\left(x^2+8x+\mathbf{16}\right)+y^2=-7+\mathbf{16}$$

Step 3: Factor the perfect square trinomial.

$$(x+4)^2+y^2=9$$

Step 4: Write in standard form.

$$[x-(-4)]^2+[y-0]^2=3^2$$

Thus, the above equation represents a circle that has its center at $(-4,0)$ and radius 3.

19. $2x^2+2y^2-16x+12y=0$

Solution

Step 1: The coefficients of x^2 *and* y^2 *must be one, before the completing the square process can be started. Divide each side of the equation by 2.*

$$x^2+y^2-8x+6y=0$$

Step 2: Regroup

$$\left(x^2-8x\right)+\left(y^2+6y\right)=0$$

Step 3: Complete the square for x, by adding $\left(-\frac{8}{2}\right)^2=16$ *to each sides of the equation.*

$$\left(x^2-8x+\mathbf{16}\right)+\left(y^2+6y\right)=\mathbf{16}$$

Step 4: Complete the square for y, by adding $\left(\frac{6}{2}\right)^2=9$ *to each side of the equation.*

$$\left(x^2-8x+\mathbf{16}\right)+\left(y^2+6y+\mathbf{9}\right)=\mathbf{16}+\mathbf{9}$$

Step 5: Factor the perfect square trinomials.

$$(x-4)^2+(y+3)^2=25$$

Step 6: Write in standard form.

$$[x-4]^2+[y-(-3)]^2=5^2$$

Thus, the above equation represents a circle that has center at $(4,-3)$ and radius 5.

20. Find the standard equation of a circle containing the point $(3,-2)$, with center at $(-1,2)$.

Solution

From the standard form of the equation of a circle

$$(x-h)^2+(y-k)^2=r^2 \qquad (1)$$

solve for r to get

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

Use $x=3, y=-2, h=-1, k=2$.

$$r=\sqrt{[3-(-1)]^2+[-2-2]^2}$$

$$r=\sqrt{16+16}=\sqrt{32}$$

Use equation (1) with $h=-1, k=2, r=\sqrt{32}$,

$$[x-(-1)]^2+[y-2]^2=\left[\sqrt{32}\right]^2$$

$$(x+1)^2+(y-2)^2=32$$

Relations and Functions

Which of the following relations are functions.

21. $G=\{(-4,1),(-3,2),(2,5),(-5,4)\}$

Solution

None of the ordered pairs have the same first coordinates. Thus, the set of ordered pairs is a **function.**

22. $H=\{(-1,2),(3,2),(1,3),(3,4)\}$

Solution

The set H is **not a function** as the ordered pairs $(3,2)$ and $(3,4)$ have the same first coordinates.

23. $\{(x,y) \mid 3x+y=2\}$

Solution

$3x+y=2$

$y=-3x+2$

The right side $-3x+2$ produces one value of y for each value of x. Thus, the given relation is a function.

24. $\{(x,y) \mid x=y^2+3\}$

Solution

$x=y^2+3$

$-y^2=-x+3$

$y^2=x-3$

$y=\pm\sqrt{x-3}$

For every $x>0$ there are 2 values of y. Thus, the given relation is not a function.

25. $\{(x,y) \mid y^3+3=x\}$

Solution

$y^3+3=x$

$y^3=x-3$

$y=\sqrt[3]{x-3}$

The given relation is a function.

26. $\{(x,y) \mid y=x^2-6\}$

Solution

$y=x^2-6$

The given relation is a function.

27. $\{(x,y) \mid y^3 + x^3 = 27\}$

Solution

$$y^3 + x^3 = 27$$
$$y^3 = 27 - x^3$$
$$y = \sqrt[3]{27 - x^3}$$

The given relation is a function.

Determine which of the graphs are graphs of functions.

28.

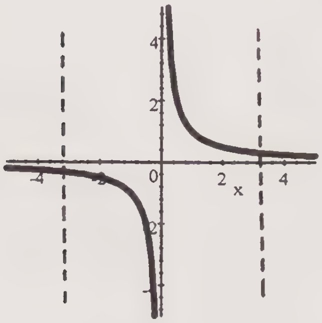


Figure 3.22

Every vertical line intersects the graph at most once. Thus, the graph is the graph of a function.

29.

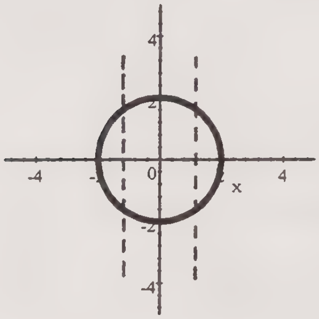


Figure 3.23

Every vertical line intersects the graph more than once. Thus, the graph is not the graph of a function.

Domain of a Function

Determine the domain of each function.

30. $f(x) = \sqrt{x^2 + x - 2}$

Solution

Since even root of negative numbers are not real numbers, the radicand must be nonnegative, that is,

$$x^2 + x - 2 \geq 0.$$

Solve this second degree polynomial to find the critical values of the inequality $x^2 + x - 2 \geq 0$.

$$(x+2)(x-1)=0$$
$$x+2=0 \quad or \quad x-1=0$$
$$x=-2 \qquad\qquad x=1$$

These critical values separate the real number line into three intervals as shown below.

	$-\infty < x \leq -2$	$-2 \leq x \leq 1$	$1 \leq x < \infty$
$x+2$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
$(x+2)(x-1) \geq 0$	$+$	$-$	$+$
Test numbers	-3	0	3

Thus, the domain $= \{x \mid -\infty < x \leq -2 \text{ or } 1 \leq x < \infty\}$.

31. $f(x) = \dfrac{3}{\sqrt{2-x}}$

Solution

The radicand must be nonnegative, that is

$$2-x \geq 0$$
$$2 \geq x$$
$$x \leq 2$$

Also, the real number 2 must be excluded from the domain since it makes the denominator zero. Therefore,

Domain $= \{x \mid x < 2\}$.

32. $f(x) = \dfrac{2}{x^3 - 7x^2 + 12x}$

Solution

Since division by zero is not defined, the domain of f excludes those values of x that causes the denominator to be zero.

Thus,

$$x^3 - 7x^2 + 12x \neq 0$$
$$x\left(x^2 - 7x + 12\right) \neq 0$$
$$x(x-4)(x-3) \neq 0$$
$$x \neq 0 \quad or \quad x \neq 4 \quad or \quad x \neq 3$$

The domain $= \{x \mid x \neq 0 \text{ or } x \neq 4 \text{ or } x \neq 3\}$.

33. $f(x) = \sqrt{\dfrac{x-1}{x+2}}$

Solution

The radicand must be nonnegative, that is,

$$\frac{x-1}{x+2} \geq 0$$

To solve a rational inequality, set the numerator and denominator equal to zero to find the critical values of the inequality.

$$x-1=0 \quad or \quad x+2=0$$
$$x=1 \qquad\qquad x=-2$$

These critical values separate the real number line into three intervals, as shown below.

	$-\infty < x < -2$	$-2 \le x \le 1$	$1 \le x < \infty$
$x-1$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$\dfrac{x-1}{x+2} \ge 0$	$+$	$-$	$+$
Test numbers	-3	0	3

The domain $= \{x \mid -\infty < x < -2 \quad \text{or} \quad 1 \le x < \infty\}$.

34. $h(x) = \dfrac{\sqrt{9-x^2}}{3+x}$

Solution

Keep in mind that domain of $h(x)$ is the intersection of the domains of the numerator and denominator.

Domain of the Numerator

$$9 - x^2 \ge 0$$
$$(3-x)(3+x) = 0$$
$$3 - x = 0 \quad \text{or} \quad 3 + x = 0$$
$$x = 3 \qquad\qquad x = -3$$

	$-\infty < x \le -3$	$-3 \le x \le 3$	$3 \le x < \infty$
$3-x$	$+$	$+$	$-$
$3+x$	$-$	$+$	$+$
$9-x^2 \ge 0$	$-$	$+$	$-$
Test numbers	-4	0	4

Domain of the numerator $= \{x/ -3 \le x \le 3\}$.

Domain of the Denominator

$$3 + x \ne 0$$
$$x \ne -3$$

Domain of the denominator $= \{x \mid x \ne -3\}$.

Domain of $h(x) = \{x/ -3 < x \le 3\}$.

35. Given $f(x) = \dfrac{x^2+1}{x^2+x-2}$,

a. *Find $f(x)$ if $x = -1$.*

b. *Is the point $\left(-3, \frac{5}{2}\right)$ on the graph of f?*

Solution

a. To find $f(x)$ when $x = -1$ is to find $f(-1)$. Thus,

$$f(-1) = \frac{(-1)^2 + 1}{(-1)^2 + (-1) - 2} = \frac{1+1}{1-1-2} = -1$$

$$f(-1) = -1$$

b. To show that the point $\left(-3, \frac{5}{2}\right)$ is on the graph of f, find the value of f at $x = -3$. If the value of f equals $\frac{5}{2}$, then it can be said that the point is on the graph of f.

$$\begin{aligned} f(x) &= \frac{x^2+1}{x^2+x-2} \\ ? &= \frac{(-3)^2+1}{(-3)^2-3-2} \\ ? &= \frac{9+1}{9-5} \\ ? &= \frac{10}{4} \\ \frac{5}{2} &= \frac{5}{2} \end{aligned}$$

Thus, the point $\left(-3, \frac{5}{2}\right)$ is on the graph of f.

36. Given $f(x) = \frac{18}{x^2+2}$, find

a. $f(-3)$
b. $f(0)$

Solution

a. $f(-3) = \frac{18}{(-3)^2+2} = \frac{18}{11}$

b. $f(0) = \frac{18}{2} = 9$

37. Given $g(x) = \frac{10}{x^2+2}$, find

a. The value of x if $g(x) = 2$.
b. The domain of g.

Solution

a. $g(x) = \frac{10}{x^2+2}$

$2 = \frac{10}{x^2+2}$ • Substitute g(x) =2.

$2x^2+4=10$ • Multiply each side by x^2+2.
$2x^2=6$ • Subtract 4 from each side.
$x^2=3$ • Divide each side by 2.
$x=\pm\sqrt{3}$ • Solve for x.

The value of $x = \pm\sqrt{3}$, when $g(x) = 2$.

b. To find the domain of g it is necessary that the denominator, $x^2+2 \neq 0$. It is worth noting that x^2+2 is always positive. Thus,

Domain $= \{x \mid x \text{ is all real numbers}\}$.

38. Given $f(x) = \frac{\sqrt{3x+7}}{|x-2|}$, find $f(-2)$.

Solution

$$\begin{aligned} f(x) &= \frac{\sqrt{3x+7}}{|x-2|} \\ f(-2) &= \frac{\sqrt{3(-2)+7}}{|-2-2|} \end{aligned}$$

$$f(-2) = \frac{\sqrt{1}}{|-4|} = \frac{1}{4}$$

Symmetry

Test each equation for symmetry.

39. $xy = 2$

Solution

a. Test for symmetry with respect to the **x-axis.**

Replace y with $-y$.
$x(-y) = 2$
$-xy = 2$

This is not equivalent to the original equation, thus the graph of $xy = 2$ is *not* symmetric with respect to the $x - axis$.

b. Test for symmetry with respect to the **y-axis.**

Replace x with $-x$.
$(-x)y = 2$
$-xy = 2$

This is not equivalent to the original equation, thus the graph of $xy = 2$ is *not* symmetric with respect to the $y - axis$.

c. Test for symmetry with respect to the **origin.**

Replace x with $-x$ and y with $-y$.
$(-x)(-y) = 2$
$xy = 2$

This is equivalent to the original equation. Thus, $xy = 2$ *is* symmetric with respect to the origin.

40. $x^2 + y^2 = 4$

Solution

a. Test for symmetry with respect to the **x-axis.**

Replace y with y.
$x^2 + (-y)^2 = 4$
$x^2 + y^2 = 4$

This is equivalent to the original equation. Thus, the graph of $x^2 + y^2 = 4$ *is* symmetric with respect to the $x - axis$.

b. Test for symmetry with respect to the **y-axis.**

Replace x with $-x$.
$(-x)^2 + y^2 = 4$
$x^2 + y^2 = 4$

This is equivalent to the original equation. Thus, the graph, the circle of radius 2 *is* symmetric with respect to the $y - axis$.

c. Test for symmetry with respect to the **origin.**

Replace x with $-x$ and y with $-y$.
$(-x)^2 + (-y)^2 = 4$
$x^2 + y^2 = 4$

This is equivalent to the original equation. Thus, the circle *is* symmetric with respect to the $origin$.

41. $y = |x| - 2$

Solution

a. Test for symmetry with respect to the **x-axis.**

Replace y with $-y$.
$-y = |x| - 2$
This is not equivalent to the original equation. Thus, the graph of $y = |x| - 2$ is *not* symmetric with respect to the $x - axis$.

b. Test for symmetry with respect to the **y-axis.**
Replace x with $-x$.
$y = |-x| - 2$
$y = |x| - 2$
This is equivalent to the original equation. Thus, the graph of $y = |x| - 2$ *is* symmetric with respect to the $y - axis$.

c. Test for symmetry with respect to the **origin.**
Replace x with $-x$ and y with $-y$.
$-y = |-x| - 2$
$-y = |x| - 2$
This is not equivalent to the original equation. Thus, the graph of $y = |x| - 2$ is *not* symmetric with respect to the $origin$.

Even and Odd Functions

Determine whether the given function is an even function, an odd function, or neither.

42. $f(x) = 4x^3 - 5$
Solution
Replace x with $-x$ and simplify.
$$f(-x) = 4(-x)^3 - 5 = -4x^3 - 5$$
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, we can conclude that this function is neither odd nor even.

43. $g(x) = 2x^3 - x$
Solution
Replace x with $-x$ and simplify.
$$g(-x) = 2(-x)^3 - (-x) = -2x^3 + x = -\left(2x^3 - x\right) = -g(x)$$
Since $g(-x) = -g(x)$, we can conclude that this function is an odd function.

44. $f(x) = \sqrt[3]{2x^2 - 1}$
Solution
Replace x with $-x$ and simplify.
$$f(-x) = \sqrt[3]{2(-x)^2 - 1} = \sqrt[3]{2x^2 - 1}$$
Since $f(x) = f(-x)$, we can conclude that this function is an even function.

Graphing Techniques

Sketch the graph of each function using the graphing techniques.

45. $f(x) = x^2 + 2$
Solution
The graph of $f(x) = x^2 + 2$ is the graph of $f(x) = x^2$ shifted vertically upward 2 units.

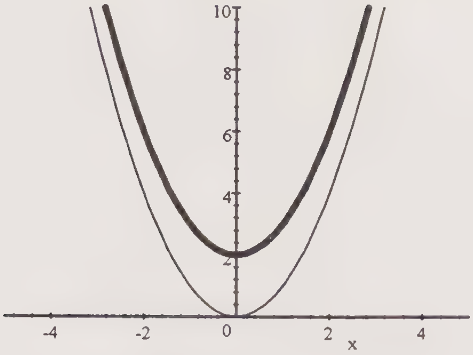


Figure 3.24

46. $f(x) = |x| - 3$

Solution

The graph of $f(x) = |x| - 3$ is the graph of $f(x) = |x|$ shifted vertically downward 3 units.

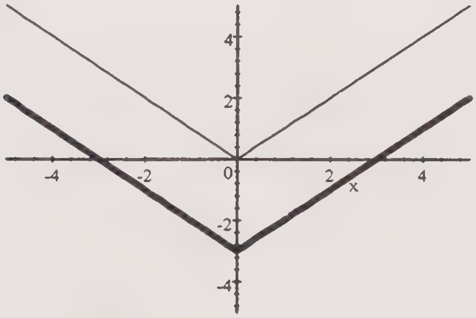


Figure 3.25

47. $f(x) = (x-1)^3$

Solution

The graph of $f(x) = (x-1)^3$ is the graph of $f(x) = x^3$ shifted horizontally to the right 1 unit.

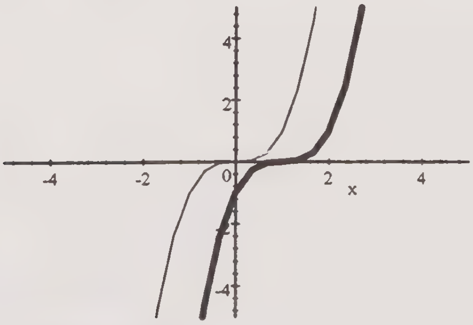


Figure 3.26

48. $f(x) = \sqrt{x+2}$

Solution

The graph of $f(x) = \sqrt{x+2}$ is the graph of $f(x) = \sqrt{x}$ shifted horizontally to the left 2 units.

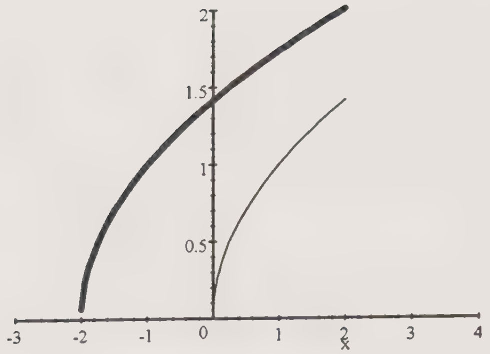


Figure 3.27

49. $f(x) = \frac{1}{2}|x|$

Solution

The graph of $f(x) = \frac{1}{2}|x|$ is the graph of $f(x) = |x|$, shrunk by a factor of $\frac{1}{2}$.

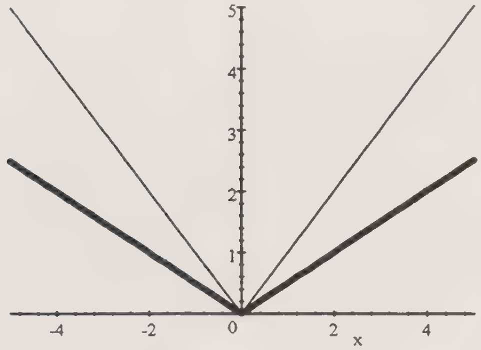


Figure 3.28

50. $f(x) = 3x^2$

Solution

The graph of $f(x) = 3x^2$ is the graph of $f(x) = x^2$, stretched by a factor of 3.

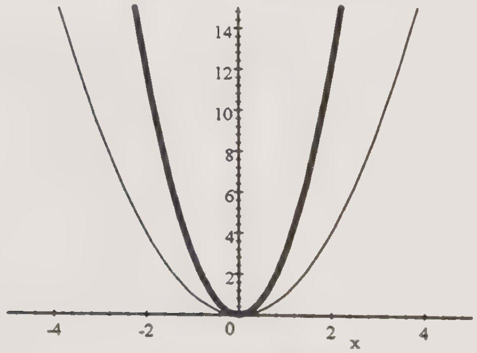


Figure 3.29

51. $f(x) = \sqrt{2-x}$

Solution

Use the following steps to get the graph of $f(x) = \sqrt{2-x}$.

Step 1: graph $f(x) = \sqrt{x}$.

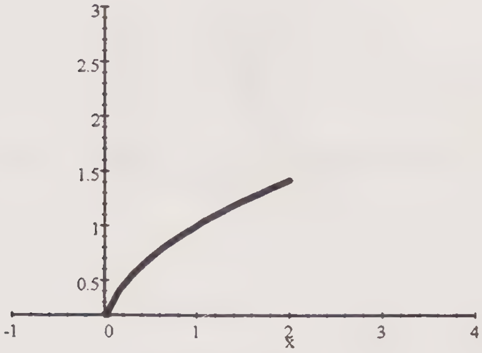


Figure 3.30

Step 2: graph $f(x) = \sqrt{2+x}$; *horizontal shift to the left* 2 units.

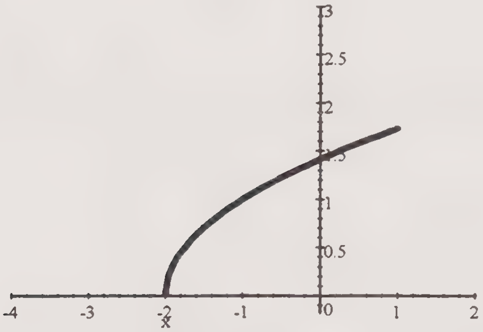


Figure 3.31

Step 3: graph $f(x) = \sqrt{2-x}$; *reflection about y-axis.*

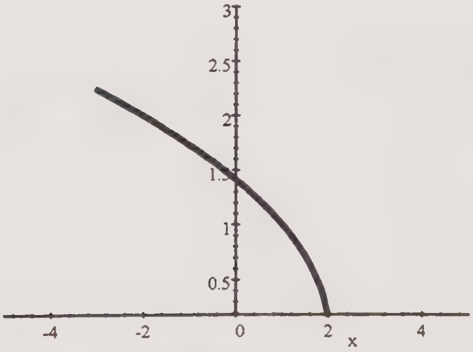


Figure 3.32

52. $f(x) = -|x-2|$

Solution

Use the following steps to get the graph of $f(x) = -|x-2|$.

Step 1: graph $f(x) = |x|$.

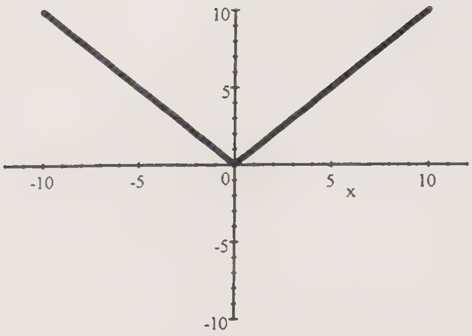


Figure 3.33

Step 2: graph $f(x) = |x-2|$; *horizontal shift to the right 2 units.*

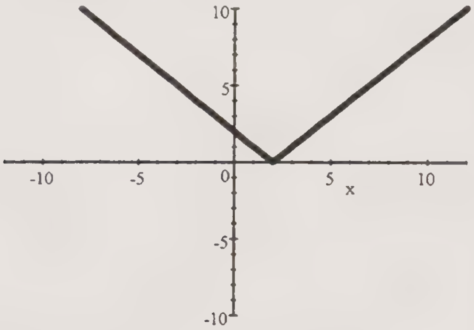


Figure 3.34

Step 3: $f(x) = -|x-2|$; *reflection about the x-axis.*

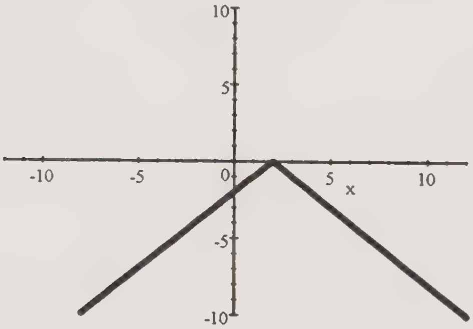


Figure 3.35

53. $f(x) = \dfrac{3}{x-1} + 2$

Solution

Use the following steps to get the graph.

Step 1: graph $f(x) = \dfrac{1}{x}$

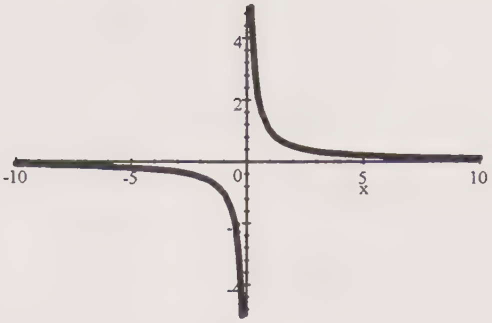


Figure 3.36

Step 2: graph $f(x) = \dfrac{3}{x}$*; vertical stretch of the graph* $f(x) = \dfrac{1}{x}$.

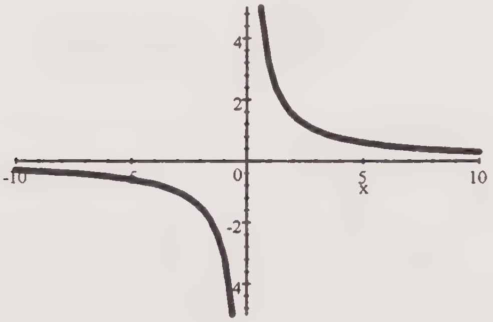


Figure 3.37

Step 3: graph $f(x) = \dfrac{3}{x-2}$*; horizontal shift to the right 2 units.*

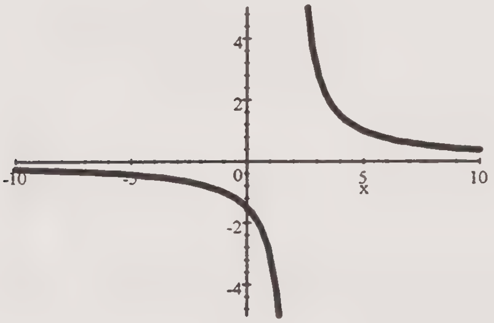


Figure 3.38

Step 4: graph $f(x) = \dfrac{3}{x-2} + 2$*; vertical shift upward 2 units.*

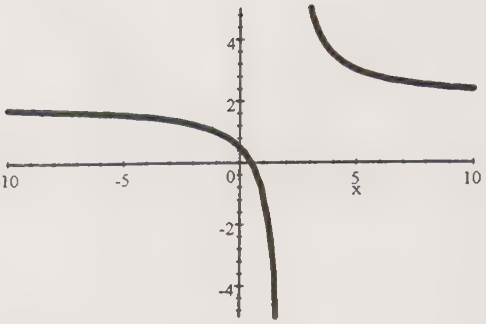


Figure 3.39

54. $f(x) = 1 - (x+2)^3$

Solution

Use the following steps to get the graph of $f(x) = 1 - (x+2)^3$.

Step 1: graph $f(x) = x^3$.

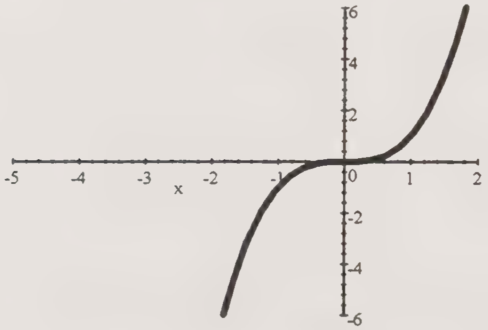


Figure 3.40

Step 2: graph $f(x) = (x+2)^3$ *; horizontal shift to the left 2 units.*

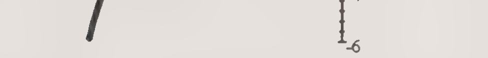


Figure 3.41

Step 3: graph $f(x) = -(x+2)^3$ *; reflection about x-axis.*

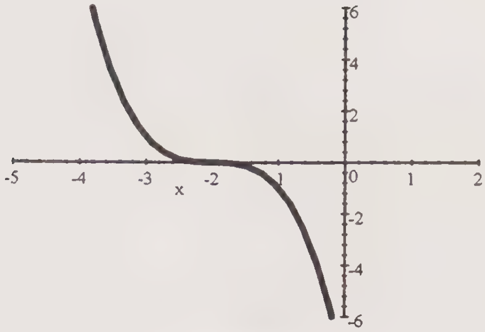


Figure 3.42

Step 4: graph $f(x) = 1 - (x+2)^3$*; vertical shift upward 1 unit.*

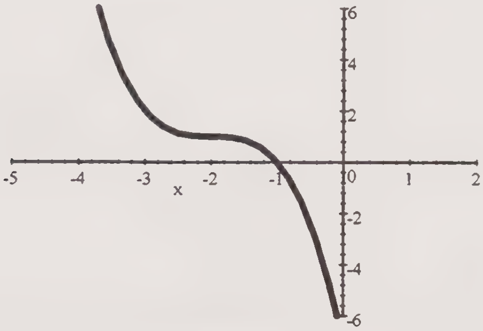


Figure 3.43

55. $f(x) = \sqrt{x-1} + 1$

Solution

Use the following steps to get the graph of f.

Step 1: graph $f(x) = \sqrt{x}$.

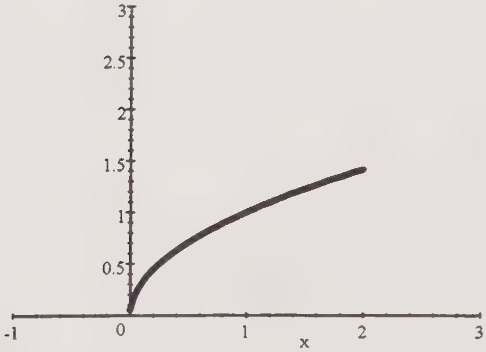


Figure 3.44

Step 2: graph $f(x) = \sqrt{x-1}$*; horizontal shift to the right 1 unit.*

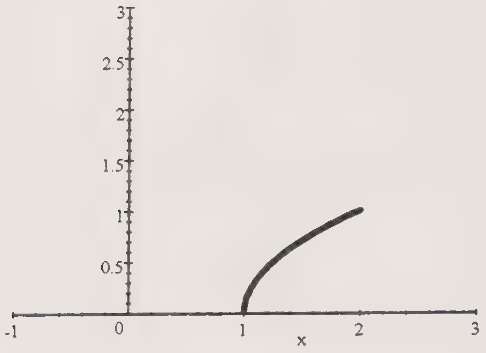


Figure 3.45

Step 3: graph $f(x) = \sqrt{x-1} + 1$*; vertical shift upward 1 units.*

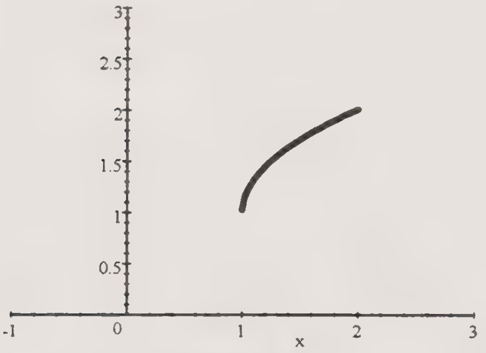


Figure 3.46

56. $g(x) = \sqrt{-x-1} + 2$

Solution

Use the following steps to get the graph of $g(x) = \sqrt{-x-1} + 2$.

Step 1: graph $g(x) = \sqrt{x}$.

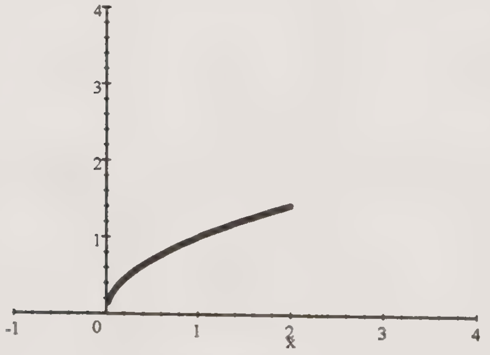


Figure 3.47

Step 2: graph $g(x) = \sqrt{x-1}$.

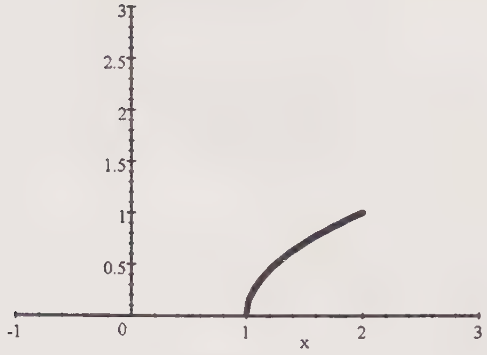


Figure 3.48

Step 3: graph $g(x) = \sqrt{-x-1}$; *reflection about y-axis.*

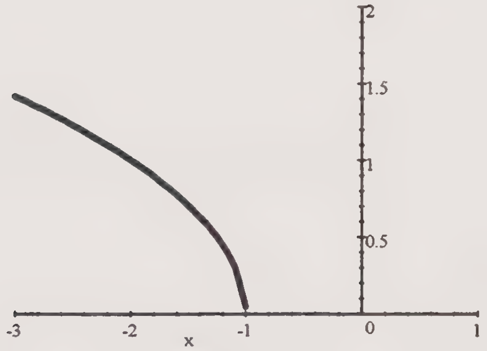


Figure 3.49

Step 4: graph $g(x) = \sqrt{-x-1} + 2$; *vertical shift upward 2 units.*

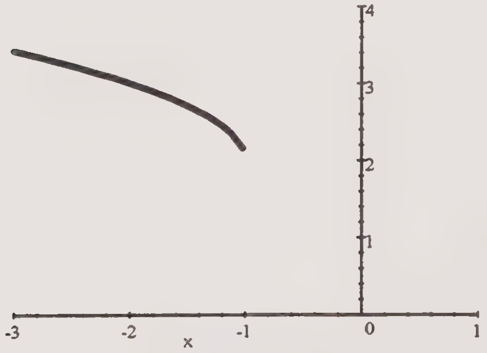


Figure 3.50

PRACTICE TEST

1. Find the distance between the points $(-2,-1)$ and $(-3,0)$; Also find the midpoint of the line segment connecting the above two points.

2. Determine the x and y intercepts of $y^2 = x^2 + 9$.

3. Find the center and radius of the circle that has the general form

$$x^2 + y^2 - 6x + 10y + 18 = 0$$

4. Determine the center and radius of the circle $x^2 + (y+4)^2 = 36$.

Test each equation for symmetry.

5. $y = \dfrac{2|x|}{x}$

6. $y = x^2 + 5$

Graph using graphing techniques.

7. $y = -2\left(x - 1\right)^2$

8. $y = \sqrt{x + 1} - 2$

Identify whether the given function is an even function, an odd function, or neither.

9. $f(x) = |x - 2|$

10. $f(x) = \dfrac{|x|}{2x}$

11. Find all the points with $x - coordinate$ of -1 whose distance from the point $(1, 4)$ is 2.

12. Determine the equation of a circle in standard form with center $(-1, 0)$ and radius -4.

13. Determine the $x-$ and $y - intercepts$, and then graph the equation $2x + y = 3$.

Which of the following equations define(s) y as a function of x. If your answer is no explain why.

14. $4x + 3y = 12$

15. $x^2 - 4y^2 = 4$

16. $x^2 - 4y = 4$

Determine the domain of each function.

17. $f(x) = 3x^2 - 5x + 1$

18. $f(x) = \sqrt{x^2 + 2x - 24}$

CHAPTER FOUR

Functions and Graphs Part II

4.1 The Algebra of Functions
4.2 Inverse Functions
4.3 Variation and Applications

4.1 THE ALGEBRA of FUNCTIONS

Operations on Functions

Given two functions f and g which are defined for all values of x, we define the following functions.

Sum	$(f+g)(x) = f(x) + g(x)$
Difference	$(f-g)(x) = f(x) - g(x)$
Product	$(fg)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad g(x) \neq 0$

Domains

For functions f and g , the domain of $f+g$, $f-g$, and fg is the set of all real numbers in the intersection of the domains of f and g. The domain of $\frac{f}{g}$ is the set of all real numbers in the intersection of the domains of f and g for which $g(x) \neq 0$.

Example 1 **Operations on Functions**

Given $f(x) = \sqrt{x-5}$ and $g(x) = \sqrt{x+3}$

Find the following and determine the domain in each case.

a. $(f+g)(x)$ **b.** $(f-g)(x)$ **c.** $(f \cdot g)(x)$ **d.** $\left(\frac{g}{f}\right)(x)$

Solution

a. $(f+g)(x) = f(x) + g(x) = \sqrt{x-5} + \sqrt{x+3}$

b. $(f-g)(x) = f(x) - g(x) = \sqrt{x-5} - \sqrt{x+3}$

c. $(f \cdot g)(x) = f(x) \cdot g(x) = \left(\sqrt{x-5}\right)\left(\sqrt{x+3}\right)$

d. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{x+3}}{\sqrt{x-5}}$

The domain of $f = \{x \mid x \geq 5\}$; the domain of $g = \{x \mid x \geq -3\}$. The intersection of the domains of f and g is the set of all numbers such that $x \geq 5$. Therefore,

Domain $f+g =$ Domain $f-g =$ Domain $f \cdot g = \{x \mid x \geq 5\}$.

For the quotient function $\frac{g}{f}$, we must exclude from this set the number 5, because the denominator, g, will be zero when $x = 5$. Thus,

Domain $\frac{g}{f} = \{x > 5\}$.

Difference Quotient

The expression

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient** of f.

Example 2 **Determine a Difference Quotient**

Determine the difference quotient of $f(x) = -3x^2 + 2$.

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\left[-3(x+h)^2+2\right]-\left[-3x^2+2\right]}{h}\\ &= \frac{\left[-3\left(x^2+2xh+h^2\right)+2\right]-\left[-3x^2+2\right]}{h}\\ &= \frac{-3x^2-6xh-3h^2+2+3x^2-2}{h}\\ &= \frac{-6xh-3h^2}{h} = -6x-3h\end{aligned}$$

Composition of Functions

If f and g are two functions, then the **composite function** or **composition** of f and g is defined by

$$(f \circ g)(x) = f[g(x)]$$

and the **composite function** of g and f is defined by

$$(g \circ f)(x) = g[f(x)]$$

Note: In general $(f \circ g)(x) \neq (g \circ f)(x)$

Example 3 **Find Composite Functions**

If $f(x) = 2x + 1$ and $g(x) = x^2 + 2x + 1$ find,

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution

a. $(f \circ g)(x) = f[g(x)]$
$= f\left[x^2 + 2x + 1\right]$ • Substitute $g(x)$ for x in $f(x)$.
$= 2\left(x^2 + 2x + 1\right) + 1$
$= 2x^2 + 4x + 3$

b. $(g \circ f)(x) = g[f(x)]$
$= g[2x + 1]$ • Substitute $f(x)$ for x in $g(x)$.
$= (2x + 1)^2 + 2(2x + 1) + 1$
$= 4x^2 + 8x + 4$

EVALUATING COMPOSITE FUNCTIONS

To evaluate $(f \circ g)(a)$ for some constant a, we can use either of the following methods:

Method 1 First evaluate $g(a)$, then substitute this result for x in $f(x)$.
Method 2 First determine $f[g(x)]$, then substitute a for x.

Example 4 **Evaluate Composite Functions**

Evaluate $(f \circ g)(0)$, where $f(x) = 2x - 3$ and $g(x) = \frac{1}{2}x + 4$.

Solution

Method 1 $(f \circ g)(0) = f[g(0)]$

$$= f\left[\tfrac{1}{2}(0) + 4\right]$$

$$= f(4)$$
$$= 2(4) - 3 = 5$$

Method 2 $(f \circ g)(x) = f[g(x)]$

$$= f\left(\tfrac{1}{2}x + 4\right)$$

$$= 2\left(\tfrac{1}{2}x + 4\right) - 3$$

$$= x + 8 - 3 = x + 5$$
$$(f \circ g)(0) = 0+5 = 5$$

DOMAINS

The domain of $(f \circ g)$ is the set of all x in the domain of g, such that $g(x)$ is in the domain of $f(x)$. The domain of $(g \circ f)$ is the set of all x in the domain of f, such that $f(x)$ is in the domain of $g(x)$.

Example 5 **Find Composition of Functions**

Given $f(x) = \sqrt{x}$, $g(x) = x - 2$ find $f \circ g$, $g \circ f$, and the domain in each case.

Solution

$(f \circ g)(x) = f[g(x)]$
$= f(x - 2)$
$= \sqrt{x - 2}$
Domain $f \circ g = [2, \infty)$

$(g \circ f)(x) = g[f(x)]$
$= g(\sqrt{x})$
$= \sqrt{x} - 2$
Domain $g \circ f = [0, \infty)$

4.2 INVERSE FUNCTIONS

Addition and subtraction are inverse operations - starting with a number x, adding 2, and subtracting 2, gives back x as a result. Similarly, some functions are inverses of each other. For example, the functions

$$f(x) = 2x + 3 \qquad \text{and} \qquad g(x) = \frac{x-3}{2}$$

are inverses of each other in the sense that one undoes the other. To illustrate this, let $x = 5$. Then,

$$f(5) = 2(5) + 3 = 13$$

Now, evaluate $g(13)$

$$g(\mathbf{13}) = \frac{13-3}{2} = \mathbf{5}$$

Thus, $g[f(5)] = g(13) = 5$

We started with $x = 5$, then evaluated $f(5)$, and finally evaluated $g[f(5)]$, and the result was the number we started with, 5. Also, $f[g(5)] = 5$.

For these functions it can be shown that

$$f[g(x)] = x \qquad \text{and} \qquad g[f(x)] = x$$

for any value of x.

In this section we will learn how to find inverse of a function. Not all functions have inverse functions. Only functions that are *one-to-one* have inverse functions.

One-to-One Function

A function f is said to be **one-to-one** if for elements a and b from the domain of f,

$$a \neq b \qquad \text{implies} \qquad f(a) \neq f(b)$$

HORIZONTAL LINE TEST

If any *horizontal line* intersects the graph of a function in no more than one point, then the function is one-to-one.

Example 6 **Identify One-to-One Functions Represented by Graphs**

Which of the following graphs are graph(s) of One-to-One functions?

a.

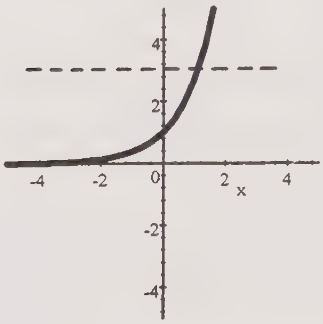


Figure 4.1

b.

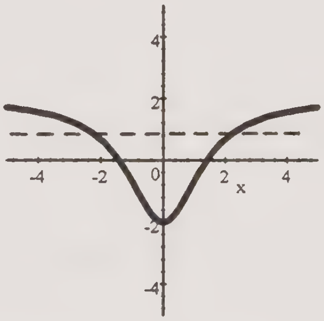


Figure 4.2

Solution

Use the horizontal line test to determine whether the graph is the graph of a one-to-one function.

a. Since no horizontal line intersects the graph more than once, the given graph is the graph of a one-to-one function.

b. Every horizontal line intersects the graph in more than one place. Therefore, the given graph is not the graph of a one-to-one function.

Example 7 **Identify One-to-One Functions Represented by Equations**

Which of the following function(s) are one-to-one?

a. $f(x) = 2x + 3$ **b.** $f(x) = x^2 + 2$

Solution

a. For this function two different values of x will always generate two different values of f.

b. Let's use $a = 2$ and $b = -2$, then $2 \neq -2$, but

$$f(2) = (2)^2 + 2 = 6$$
$$f(-2) = (-2)^2 + 2 = 6$$

Here, although $2 \neq -2$, $f(2) = f(-2)$. Therefore, by definition this is not a one-to-one function.

Inverse Functions

Let f be a one-to-one function. Then g is the **inverse function** of f if and only if

$$(f \circ g)(x) = x \qquad \text{for all } x \text{ in the domain of } g$$

and

$$(g \circ f)(x) = x \qquad \text{for all } x \text{ in the domain of } f$$

A special notation is used for inverse functions - if g is the inverse of a function f, then g is written as f^{-1} (read "f *inverse*").

CAUTION: Do not confuse the -1 in f^{-1} with a negative exponent. The symbol $f^{-1}(x)$ does not represent $\frac{1}{f(x)}$; it represents the inverse function. Remember that a function f can have an inverse function if and only if it is a one-to-one function.

Example 8 **Verify that Two Functions are Inverse Functions**

Verify that $g(x) = \sqrt[3]{\frac{x+1}{2}}$ is the inverse function of $f(x) = 2x^3 - 1$.

Solution

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right)$$
$$= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$
$$= 2\left(\frac{x+1}{2}\right) - 1$$
$$= x + 1 - 1 = x$$

$$g(f(x)) = g(2x^3 - 1)$$
$$= \sqrt[3]{\frac{2x^3 - 1 + 1}{2}}$$
$$= \sqrt[3]{\frac{2x^3}{2}} = x$$

Since both $(f \circ g)(x)$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

FINDING THE INVERSE OF A FUNCTION

A function f has an inverse function if and only if it is a one-to-one function.

To find the inverse function f^{-1} proceed as follows:

1. *Substitute y for f(x).*
2. *Interchange x and y.*
3. *Solve for y in terms of x.*
4. *Substitute $f^{-1}(x)$ for y in the result.*

Note: *The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.*

Example 9 **Find the Inverse of a Function**

Find the inverse of the function $f(x) = \dfrac{3-2x}{5}$.

Solution

$$f(x) = \frac{3-2x}{5}$$

$$y = \frac{3-2x}{5}$$ • Replace $f(x)$ by y.

$$x = \frac{3-2y}{5}$$ • Interchange x and y.

$$5x = 3 - 2y$$ • Multiply both sides by 5.

$$2y = 3 - 5x$$ • Isolate the y-term.

$$y = \frac{3-5x}{2}$$ • Solve for y.

$$f^{-1}(x) = \frac{3-5x}{2}$$ • Replace y by $f^{-1}(x)$.

Example 10 **Find the Inverse of a Function**

Find the inverse of the function $f(x) = x^2 - 2x$, $x \geq 1$. State the domain and range of f^{-1}.

Solution

The function f is one-to-one with domain $[1, \infty)$. As mentioned before,
Domain f = Range $f^{-1} = [1, \infty)$

To find f^{-1} proceed as follows using the completing the square technique.

$$f(x) = x^2 - 2x$$

$$y = x^2 - 2x$$

$$x = y^2 - 2y$$ • Interchange x and y.

$$x + 1 = y^2 - 2y + 1$$ • To complete the square add 1 to each side.

$$x + 1 = (y-1)^2$$ • Factor.

$$\pm\sqrt{x+1} = y - 1$$ • Apply the Square Root Theorem.

$1 \pm \sqrt{x+1} = y$

The range of f^{-1} is $[1, \infty)$. In order to make $1 \pm \sqrt{x+1}$ a number greater than or equal to 1 the nonnegative square root must be chosen, thus

$$f^{-1}(x) = 1 + \sqrt{x+1},\ x \geq -1$$

Range f = Domain $f^{-1} = [-1, \infty)$

Example 11 **Find the Inverse of a Function**

Find the inverse of the function $f(x) = x^2 - 4x + 1, x \geq 2$. State the domain and range of f^{-1}.

Solution

The function f is one-to-one with domain $[2, \infty)$. As mentioned before,
Domain f = Range $f^{-1} = [2, \infty)$

To find f^{-1} proceed as follows using the completing the square technique.

$$f(x) = x^2 - 4x + 1$$
$$y = x^2 - 4x + 1$$
$$x = y^2 - 4y + 1 \qquad \bullet \text{ Interchange x and y.}$$
$$x - 1 = y^2 - 4y$$
$$x - 1 + 4 = y^2 - 4y + 4 \qquad \bullet \text{ To complete the square, add 4 to each side.}$$
$$x + 3 = (y-2)^2 \qquad \bullet \text{ Factor.}$$
$$\pm\sqrt{x+3} = y - 2 \qquad \bullet \text{ Apply the Square Root Theorem.}$$
$$y = 2 \pm \sqrt{x+3}$$

The range of f^{-1} is $[2, \infty)$. In order to make $2 \pm \sqrt{x+3}$ a number greater than or equal to 2 the nonnegative square root must be chosen, thus

$$f^{-1}(x) = 2 + \sqrt{x+3},\ x \geq -3$$

Therefore, Domain f^{-1} = Range $f = [-3, \infty)$.

4.3 VARIATION AND APPLICATIONS

In many real-life applications of mathematics, it is necessary to express relationships between quantities. Such relationships are called **variation.** For example, in physics, Hooke's Law shows how the distance a spring is stretched or compressed from its natural or equilibrium length relates to the force on the spring. In chemistry, the Ideal Gas Law shows how volume, temperature, and pressure are related. There are four types of variations: direct, inverse, joint, and combined variation.

Direct Variation

y **varies directly** as x, or y is **directly proportional** to x, if there is a nonzero real number k, called the **constant of proportionality,** such that

$$y = kx$$

Example 12 Solve a Direct Variation

Hooke's Law states that the distance, d, a spring stretches varies directly as the weight, w, on the spring. A weight of 60 pounds stretches a spring 4.5 inches. If the spring is stretched 11 inches, how much weight is put on the spring?

Solution

The given relationship can be translated into the following variation equation.

$$d = kw$$

Substitute $w = 60$, $d = 4.5$ and solve for k.

$$4.5 = k(60) \quad \Longrightarrow \quad k = \frac{4.5}{60} = 0.075$$

Therefore, the specific equation is

$$d = 0.075w$$

Substitute $d = 11$ in the given equation.

$$11 = 0.075w \quad \text{and} \quad w = \frac{11}{0.075} \approx 147 \text{ pounds}$$

Inverse Variation

y **varies inversely** as x, or y is **inversely proportional** to x, if there is a nonzero real number k, such that

$$y = \frac{k}{x}$$

Example 13 Solve an Inverse Variation

The base, b, of a triangle of fixed area varies inversely as its height, h. If the base of a triangle is 6 feet when height is 5 feet, find the base when the height is 3 feet.

Solution

The given relationship can be translated into the following variation equation.

$$b = \frac{k}{h}$$

Substitute $b = 6$, $h = 5$ and solve for k.

$$6 = \frac{k}{5} \quad \Longrightarrow \quad k = 30$$

Now that we have k, the variation equation becomes

$$b = \frac{30}{h}$$

Substitute $h = 3$, the base is

$$b = \frac{30}{3} = 10 \text{ ft.}$$

Joint And Combined Variation

When a variable y is proportional to the product of two or more other variables, it is said that y **varies jointly** as these variables. Combinations of direct and/or inverse variation is called **combined variation.**

Example 14 **Solve a Combined Variation**

The electrical resistance, R , of a wire (measured in ohms) varies directly as the length, l, of the wire and inversely as the square of the diameter, d, of the wire. If the resistance of a 400 cm long wire of diameter 0.2 cm is 10 ohms, find the resistance of 1500 cm of the same type of wire with a diameter of 0.05 cm.

Solution

The given relationship can be translated into the following variation equation.

$$R = \frac{kl}{d^2}$$

Substitute $R = 10$, $d = 0.2$, and $l = 400$ in the above equation and solve for k.

$$10 = \frac{k\,(400)}{(0.2)^2} \quad \Longrightarrow 10\,(0.2)^2 = 400k$$

$k = 0.001$

Now that we have k, the variation equation becomes

$$R = \frac{0.001l}{d^2}$$

At this point substitute $l = 1500$ and $d = 0.05$.

$$R = \frac{0.001\,(1500)}{(0.05)^2} \Longrightarrow R = 600 \text{ ohms}$$

SOLVED PROBLEMS

The Algebra of Functions

For the given functions determine $f+g$, $f-g$, $f\cdot g$, *and* $\dfrac{f}{g}$ *and state their domains.*

1. $f(x)=x^2+x-6$, $g(x)=x-2$

Solution

$(f+g)(x)=f(x)+g(x)=x^2+x-6+x-2=x^2+2x-8$

Domain $f+g$ is all real numbers.

$(f-g)(x)=f(x)-g(x)=x^2+x-6-x+2=x^2-4$

Domain $f-g$ is all real numbers.

$(f\cdot g)(x)=f(x)\cdot g(x)=(x^2+x-6)(x-2)=x^3-x^2-8x+12$

Domain $f\cdot g$ is all real numbers.

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x^2+x-6}{x-2}=\frac{(x-2)(x+3)}{x-2}=x+3$$

Domain $\dfrac{f}{g}=\{x\mid x\neq 2\}$

2. $f(x)=\sqrt{x-3}$, $g(x)=\sqrt{4-x}$

Solution

$(f+g)(x)=f(x)+g(x)=\sqrt{x-3}+\sqrt{4-x}$

Domain $f+g=[3,4]$

$(f-g)(x)=f(x)-g(x)=\sqrt{x-3}-\sqrt{4-x}$

Domain $f-g=[3,4]$

$(f\cdot g)(x)=f(x)\cdot g(x)=\sqrt{x-3}\cdot\sqrt{4-x}=\sqrt{(x-3)(4-x)}$

Domain $=[3,4]$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt{x-3}}{\sqrt{4-x}}=\sqrt{\frac{x-3}{4-x}}$$

Note that 4 had to be excluded from the domain of $\dfrac{f}{g}$ since $\dfrac{f}{g}$ is not defined when $g(x)=0$.

Domain $\dfrac{f}{g}=[3,4)$

Evaluate each function, where $f(x)=2x^2-3$ and $g(x)=x+5$.

3. $(f+g)(-2)$

Solution

$$\begin{aligned}(f+g)(-2)&=f(-2)+g(-2)\\&=\left[2(\mathbf{-2})^2-3\right]+[\mathbf{-2}+5]\\&=[8-3]+[3]=8\end{aligned}$$

4. $(f-g)\left(\dfrac{1}{2}\right)$

Solution

$$\begin{aligned}(f-g)\left(\frac{1}{2}\right)&=f\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\\&=\left[2\left(\frac{1}{2}\right)^2-3\right]-\left[\frac{1}{2}+5\right]\\&=\left[2\left(\frac{1}{4}\right)-3\right]-\left[\frac{11}{2}\right]\end{aligned}$$

$$= -\frac{5}{2} - \frac{11}{2} = -8$$

5. $(fg)(-3)$

Solution

$$\begin{aligned}(fg)(-3) &= f(-3)\,g(-3)\\ &= \left[2(\mathbf{-3})^2 - 3\right][\mathbf{-3}+5]\\ &= [2(9)-3][2] = 30\end{aligned}$$

6. $\left(\frac{f}{g}\right)(-6)$

Solution

$$\begin{aligned}\left(\frac{f}{g}\right)(-6) &= \frac{f(-6)}{g(-6)}\\ &= \frac{2(\mathbf{-6})^2 - 3}{\mathbf{-6}+5}\\ &= \frac{72-3}{-1} = -69\end{aligned}$$

7. Given $f(x) = 2x - 3$, $g(x) = \frac{1}{2}x + 4$, and $h(x) = \sqrt{2x-8}$, evaluate $(f - h + g)(4)$.

Solution

$$\begin{aligned}(f-h+g)(\mathbf{4}) &= f(\mathbf{4}) - h(\mathbf{4}) + g(\mathbf{4})\\ &= [2(\mathbf{4})-3] - \sqrt{2(\mathbf{4})-8} + \left[\tfrac{1}{2}(\mathbf{4})+4\right]\\ &= [8-3] - \sqrt{8-8} + [2+4]\\ &= 11\end{aligned}$$

$$(f-h+g)(4) = 11$$

Composition of Functions

For the given functions, determine $f \circ g$ and $g \circ f$ and state their domains.

8. $f(x) = 7x + 3$, $g(x) = \dfrac{x-3}{7}$

Solution

$$\begin{aligned}(f \circ g)(x) &= f[g(x)]\\ &= f\left(\frac{x-3}{7}\right)\\ &= 7\left(\frac{\mathbf{x-3}}{\mathbf{7}}\right)+3 \qquad \bullet \text{ Substitute } g(x) \text{ for x in } f(x).\\ &= x\end{aligned}$$

Domain of $f \circ g = \{\text{all real numbers}\}$.

$$\begin{aligned}(g \circ f)(x) &= g[f(x)]\\ &= g(7x+3)\\ &= \frac{\mathbf{7x+3}-3}{7}\end{aligned}$$

$= x$

Domain $g \circ f = \{\text{all real numbers}\}$

9. $f(x) = \sqrt{x+2}$, $g(x) = x - 4$

Solution

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f(x-4) \\ &= \sqrt{\mathbf{x} - 4 + 2} \\ &= \sqrt{x-2}\end{aligned}$$

Domain $f \circ g = [2, \infty)$

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= g\left(\sqrt{x+2}\right) \\ &= \left(\sqrt{\mathbf{x}+\mathbf{2}}\right) - 4\end{aligned}$$

Domain $g \circ f = [-2, \infty)$.

10. $f(x) = \sqrt{x-4}$, $g(x) = x^2 + 4$

Solution

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f\left(x^2 + 4\right) \\ &= \sqrt{\mathbf{x}^2 + 4 - 4} \\ &= |x|\end{aligned}$$

The domain of $f \circ g$ is all real numbers in the domain of g such that $g(x)$ is in the domain of f. The domain of g is $(-\infty, \infty)$, and the domain of f is $[4, \infty)$. Since any value of x in the interval $(-\infty, \infty)$ gives us nonnegative values of g that are in the domain of f, therefore domain of $f \circ g$ is all real numbers.

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= g\left(\sqrt{x-4}\right) \\ &= \left(\sqrt{x-4}\right)^2 + 4 \\ &= x - 4 + 4 \\ &= x\end{aligned}$$

The domain of $g \circ f$ is all real numbers in the domain of f such that $f(x)$ is in the domain of g. Since the domain of f is $[4, \infty)$, we can choose values of x only from this interval. Since every value of x in this interval gives us values of f that is in the domain of g, therefore domain of $g \circ f = [4, \infty)$

Note: The domain of $g \circ f$ in problem 10 shows that we cannot always look at the equation of the composite function and determine the domain.

11. $f(x) = \sqrt[5]{x+2}$, $g(x) = x^5 - 2$

Solution

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f\left(x^5 - 2\right) \\ &= \sqrt[5]{x^5 - 2 + 2} \\ &= x\end{aligned}$$

Domain $f \circ g$ is all real numbers.

$(g \circ f)(x) = g[f(x)]$

$$\begin{aligned}&= g\left(\sqrt[5]{x+2}\right)\\ &= \left(\sqrt[5]{x+2}\right)^5 - 2\\ &= x + 2 - 2\\ &= x\end{aligned}$$

Domain $g \circ f$ is all real numbers.

Suppose $f(x) = 2x - 3$, $g(x) = \frac{1}{2}x + 4$, and $h(x) = \sqrt{2x - 8}$. *Evaluate each of the following functions.*

12. $(g \circ f)(1)$

Solution

$$\begin{aligned}(g \circ f)(1) &= g[f(1)]\\ &= g[2(1) - 3]\\ &= g(-1)\\ &= \frac{1}{2}(-1) + 4\end{aligned}$$

$$(g \circ f)(1) = \frac{7}{2}$$

13. $(h \circ g)(4)$

Solution

$$\begin{aligned}(h \circ g)(4) &= h[g(4)]\\ &= h\left[\frac{1}{2}(4) + 4\right]\\ &= h(6)\\ &= \sqrt{2(6) - 8}\\ &= \sqrt{4}\end{aligned}$$

$$(h \circ g)(4) = 2$$

14. $(f \circ (g \circ h))(6)$

Solution

$$\begin{aligned}(f \circ (g \circ h))(6) &= f[g(h(6))]\\ &= f\left[g\left(\sqrt{2(6) - 8}\right)\right]\\ &= f\left[g\left(\sqrt{4}\right)\right]\\ &= f[g(2)]\\ &= f\left[\tfrac{1}{2}(2) + 4\right]\\ &= f(5)\\ &= 2(5) - 3\end{aligned}$$

$$(f \circ (g \circ h))(6) = 7$$

15. $(h \circ g)(2) - (f \circ g)(2)$

Solution

$$\begin{aligned}(h \circ g)(\mathbf{2}) - (f \circ g)(\mathbf{2}) &= h[g(\mathbf{2})] - f[g(\mathbf{2})]\\ &= h\left[\tfrac{1}{2}(\mathbf{2}) + 4\right] - f\left[\tfrac{1}{2}(\mathbf{2}) + 4\right]\\ &= h(5) - f(5)\\ &= \sqrt{2(5) - 8} - [2(5) - 3]\\ &= \sqrt{2} - [10 - 3]\end{aligned}$$

$$(h \circ g)(2) - (f \circ g)(2) = \sqrt{2} - 7$$

Find the difference quotient of each function in lowest terms.

16. $f(x) = 3x + 5$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) + 5 - (3x+5)}{h}\\ &= \frac{3x + 3h + 5 - 3x - 5}{h}\\ &= \frac{3h}{h} = 3\end{aligned}$$

17. $f(x) = 2x^2 + 5x + 3$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 + 5(x+h) + 3 - (2x^2 + 5x + 3)}{h}\\ &= \frac{2(x^2 + 2xh + h^2) + 5x + 5h + 3 - (2x^2 + 5x + 3)}{h}\\ &= \frac{2x^2 + 4xh + 2h^2 + 5x + 5h + 3 - 2x^2 - 5x - 3}{h}\\ &= \frac{4xh + 2h^2 + 5h}{h} = 4x + 2h + 5\end{aligned}$$

Inverse Functions

Determine whether each graph represents the graph of a one-to-one function.

18.

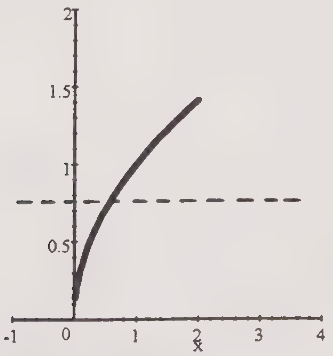


Figure 4.3

Solution

Since no horizontal line intersects the graph more than once, the graph is the graph of a one-to-one function.

19.

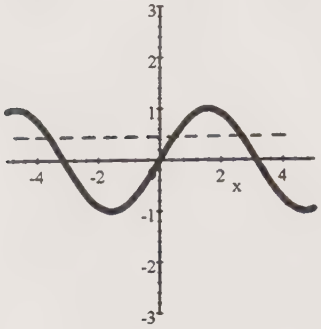


Figure 4.4

Solution

Every horizontal line intersects the graph at more than one point. Therefore, the graph is not the graph of a one-to-one function.

Find the inverse of the given functions.

20. $f(x) = 2x + 1$

Solution

$f(x) = 2x + 1$

$y = 2x + 1$ • Substitute y for f.

$x = 2y + 1$ • Interchange x and y.

$x - 1 = 2y$ • Solve for y.

$$\frac{x-1}{2} = y$$

$f^{-1}(x) = \dfrac{x-1}{2}$ • Replace y with $f^{-1}(x)$.

21. $f(x) = 3x^3 + 4$

Solution

$f(x) = 3x^3 + 4$

$y = 3x^3 + 4$ • Replace f(x) with y.

$x = 3y^3 + 4$ • Interchange x and y.

$x - 4 = 3y^3$ • Solve for y.

$$\sqrt[3]{\frac{x-4}{3}} = y$$

$f^{-1}(x) = \sqrt[3]{\dfrac{x-4}{3}}$ • Replace y with $f^{-1}(x)$.

22. $f(x) = \dfrac{x-5}{2}$

Solution

$f(x) = \dfrac{x-5}{2}$

$y = \dfrac{x-5}{2}$ • Replace $f(x)$ with y.

$x = \frac{y-5}{2}$ • Interchange x and y.

$2x = 2\left(\frac{y-5}{2}\right)$ • Multiply both sides by 2.

$2x = y - 5$

$2x + 5 = y$ • Solve for y.

$f^{-1}(x) = 2x + 5$ • Replace y with $f^{-1}(x)$.

Find the inverse of f. State the domain and range of both f and f^{-1}.

23. $f(x) = 2x^2 - 1, \quad x \geq 0$

Solution

Domain f = Range $f^{-1} = [0, \infty)$

To find f^{-1} proceed as follows:

$$f(x) = 2x^2 - 1$$
$$y = 2x^2 - 1$$
$$x = 2y^2 - 1$$

$$\frac{x+1}{2} = y^2$$

$$y = \pm\sqrt{\frac{x+1}{2}}$$

The range of f^{-1} is $\{y \mid y \geq 0\}$. In order to make $\sqrt{\frac{x+1}{2}}$ a real number greater than or equal to zero, only the nonnegative square root must be considered. Thus, f^{-1} is given by

$$f^{-1}(x) = \sqrt{\frac{x+1}{2}}, \qquad x \geq -1$$

Range f = Domain $f^{-1} = [-1, \infty)$.

24. $f(x) = \sqrt{x} + 3, \qquad x \geq 0$

Solution

Figure 4.5 illustrates the graph of $f(x) = \sqrt{x} + 3$, $x \geq 0$. In the definition of f, the domain was given as $[0, \infty)$. The Range of f is $[3, \infty)$. As mentioned before,

Domain f =Range $f^{-1} = [0, \infty)$

Range f = Domain $f^{-1} = [3, \infty)$.

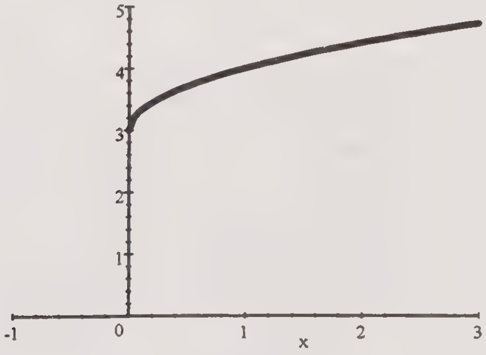


Figure 4.5

To find f^{-1}proceed as follows:

$$f(x) = \sqrt{x} + 3$$
$$y = \sqrt{x} + 3$$

$$\begin{aligned} x &= \sqrt{y}+3 \\ x-3 &= \sqrt{y} \\ (x-3)^2 &= y \end{aligned}$$

$$f^{-1}(x) = (x-3)^2, \qquad x \geq 3$$

25. $f(x) = \sqrt{4-x^2}, \qquad 0 \leq x \leq 2$

Solution

Figure 4.6 illustrates the graph of $f(x) = \sqrt{4-x^2}$, $0 \leq x \leq 2$. In the definition of f the domain was given as $[0,2]$. The range of f is $[0,2]$. As mentioned before,

Domain f = Range $f^{-1} = [0,2]$
Range f = Domain $f^{-1} = [0,2]$

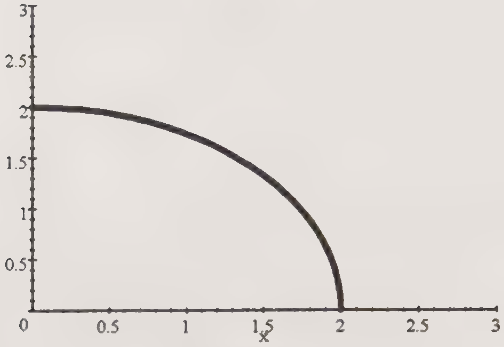


Figure 4.6

To find f^{-1} proceed as follows:

$f(x) = \sqrt{4-x^2}$
$y = \sqrt{4-x^2}$
$x = \sqrt{4-y^2}$ • Interchange x and y.
$x^2 = 4-y^2$ • Square each side.
$y^2 = 4-x^2$ • Move the y^2 term to the left and x^2 to the right.
$y = \pm\sqrt{4-x^2}$ • Apply the square root theorem.

The range of f^{-1} is $\{y \mid 0 \leq y \leq 2\}$. In order to make $\sqrt{4-x^2}$ a real number between zero and two, only the nonnegative square root must be considered. Thus f^{-1} is given by

$$f^{-1}(x) = \sqrt{4-x^2}$$

26. $f(x) = (x-4)^2, x \leq 4$

Solution

Figure 4.7 shows the graph of $f(x) = (x-4)^2$, $x \leq 4$. In the definition of f above, the domain was given as $(-\infty, 4]$. The range of f is $[0,\infty)$. As mentioned before,

Domain f = Range $f^{-1} = (-\infty, 4]$
Range f = Domain $f^{-1} = [0,\infty)$.

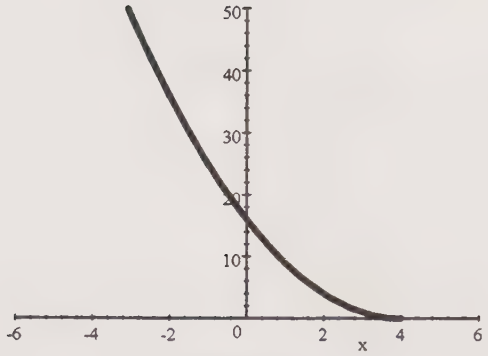


Figure 4.7

To find f^{-1} proceed as follows:

$$\begin{aligned} f(x) &= (x-4)^2 \\ y &= (x-4)^2 \\ x &= (y-4)^2 \\ \pm\sqrt{x} &= y-4 \\ \pm\sqrt{x}+4 &= y \end{aligned}$$

The range of f^{-1} is $\{y \mid y \le 4\}$. In order to make $\pm\sqrt{x}+4$ a real number less than or equal to 4, the negative square root must be considered. Thus, f^{-1} is given by

$$f^{-1}(x) = -\sqrt{x}+4$$

27. $f(x) = x^2 + 3x,\ x \ge -1$

Solution

The function f is one-to-one with the domain $[-1,\infty)$. As mentioned before, Domain f = Range $f^{-1} = [-1,\infty)$

To find f^{-1} proceed as follows using the completing the square technique.

$$\begin{aligned} f(x) &= x^2+3x \\ y &= x^2+3x \\ x &= y^2+3y \\ x+\tfrac{9}{4} &= y^2+3y+\tfrac{9}{4} \\ x+\tfrac{9}{4} &= \left(y+\tfrac{3}{2}\right)^2 \\ \pm\sqrt{x+\tfrac{9}{4}} &= \left(y+\tfrac{3}{2}\right) \\ -\tfrac{3}{2}\pm\sqrt{x+\tfrac{9}{4}} &= y \end{aligned}$$

The range of f^{-1} is $[-1,\infty)$. In order to make $-\frac{3}{2}\pm\sqrt{x+\frac{9}{4}}$ a number greater than or equal to -1 the nonnegative square root must be chosen, thus

$$f^{-1}(x) = -\tfrac{3}{2}+\sqrt{x+\tfrac{9}{4}},\ \ x \ge -\tfrac{9}{4}$$

Range f = Domain $f^{-1} = \left[-\frac{9}{4},\infty\right)$

Verify that the functions f and g are inverses of each other.

28. $f(x) = x^3-2,\ g(x) = \sqrt[3]{x+2}$

Solution

Show that $f(g(x)) = x$ and $g(f(x)) = x$.

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{x+2}\right) = \left(\sqrt[3]{x+2}\right)^3 - 2 = x+2-2 = x \\ g(f(x)) &= g\left(x^3-2\right) = \sqrt[3]{x^3-2+2} = \sqrt[3]{x^3} = x \end{aligned}$$

Therefore, functions f and g are inverses of each other. That is, f is inverse of g

and g is inverse of f.

29. $f(x) = 2x - 3,\ g(x) = \dfrac{x+3}{2}$

Solution

$$f(g(x)) = f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

$$g(f(x)) = g(2x-3) = \frac{2x-3+3}{2} = \frac{2x}{2} = x$$

Since $f(g(x)) = g(f(x)) = x$, the functions f and g are inverses of each other.

30. $f(x) = \sqrt[5]{x-4},\ g(x) = x^5 + 4$

Solution

$$f(g(x)) = f\left(x^5+4\right) = \sqrt[5]{x^5+4-4} = x$$

$$g(f(x)) = g\left(\sqrt[5]{x-4}\right) = \left(\sqrt[5]{x-4}\right)^5 + 4 = x - 4 + 4 = x$$

Since $f(g(x)) = g(f(x)) = x$, the functions f and g are inverses of each other.

31. $f(x) = (x-3)^3,\ g(x) = \sqrt[3]{x} + 3$

Solution

$$f(g(x)) = f(\sqrt[3]{x}+3) = (\sqrt[3]{x}+3-3)^3 = x$$

$$g(f(x)) = g\left((x-3)^3\right) = \sqrt[3]{(x-3)^3} + 3 = x - 3 + 3 = x$$

Since $f(g(x)) = g(f(x)) = x$, the functions f and g are inverses of each other.

Direct Variation

32. z varies directly as x, and $z = 45$ when $x = 25$, find z when $x = 30$.

Solution

Write an equation that relates z and x. Since z varies directly as x, our equation is

$$z = kx$$

It is given that $z = 45$ when $x = 25$, so

$$45 = k(25) \quad \text{and} \quad k = \frac{9}{5}$$

Therefore, the specific equation is

$$z = \frac{9}{5}x$$

When x is 30, value of z is given by

$$z = \frac{9}{5}(30) = 54$$

33. The time (t) it takes for a pendulum to make one complete swing (its period), varies directly as the square root of its length (l). If a pendulum 1.2 feet long has a period of 0.5 seconds, find the period of a pendulum 5 feet long.

Solution

The given relationship can be translated into the following variation equation

$$t = k\sqrt{l}$$

Substitute $l = 1.2$ and $t = 0.5$ and solve for k.

$$0.5 = k\sqrt{1.2} \quad \Longrightarrow \quad k = \frac{0.5}{\sqrt{1.2}}$$

Therefore, the specific equation is

$$t = \frac{0.5}{\sqrt{1.2}} l$$

Now substitute $l = 5$ in the variation equation to get

$$t = \frac{0.5}{\sqrt{1.2}} \left(\sqrt{5}\right) \approx 1.02 \text{ seconds}$$

Inverse Variation

34. If y varies inversely as x, and $y = 24$ when $x = 15$; find y when $x = 25$.

Solution

The given relationship can be translated into the following variation equation

$$y = \frac{k}{x}$$

Substitute $y = 24$ and $x = 15$ and solve for k.

$$24 = \frac{k}{15} \quad \Longrightarrow \quad k = 360$$

Therefore, the specific equation is

$$y = \frac{360}{x}$$

Now substitute $x = 25$ into the variation equation.

$$y = \frac{360}{25} = 14.4$$

35. Boyle's law states that the pressure, P, of a gas at a constant temperature varies inversely as the volume, V, it occupies. The pressure of a gas in a balloon with volume 25 cubic inches is 12 pounds per square inch. Find the pressure of the gas if the volume of the gas is increased to 40 cubic inches.

Solution

The given relationship can be translated into the following variation equation

$$P = \frac{k}{V}$$

Substitute $P = 12$ and $V = 25$ and solve for k.

$$12 = \frac{k}{25} \quad \text{and} \quad k = 300$$

Therefore, the specific equation is

$$P = \frac{300}{V}$$

Now substitute $V = 40$ in the variation equation.

$$P = \frac{300}{40} = 7.5 \text{ pounds per square inch.}$$

Joint Variation

36. The volume V of a right circular cone varies jointly as the square of the radius r and the height h. Discuss what happens to V when

a. r is doubled.
b. h is doubled.
c. both r and h are doubled.

Solution

The equation is

$$V = kr^2h$$

a. If r is doubled

$$V = k\left(2r\right)^2 h$$
$$V = 4kr^2h$$

Thus, the volume is 4 times larger.

b. If h is doubled

$$V = kr^2\left(2h\right)$$
$$V = 2kr^2h$$

Thus, the volume is two times larger.

c. If both r and h are doubled

$$V = k\left(2r\right)^2\left(2h\right)$$
$$V = 8kr^2h$$

Thus, the volume is eight times larger.

Combined Variation

37. The owners of a bagel shop find that their weekly sales of bagels, S, varies directly with their advertising budget, B, and inversely with their bagel price, p. When the advertising budget of the shop is \$250, and the price is \$1.50, they sell 3000 bagels per week. What is the expected sales of the shop if the advertising budget is \$500, and the price is \$2.00?

Solution

The general form of the equation is

$$S = \frac{kB}{p}$$

Now, find k using the known values.

$$3000 = \frac{k\left(250\right)}{1.50}$$

$$4500 = 250k$$
$$k = 18$$

Therefore, the specific equation is

$$S = \frac{18B}{p}$$

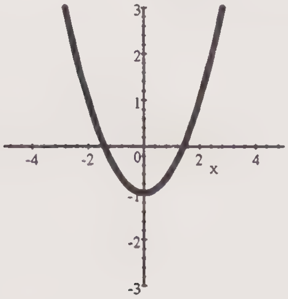
To find the expected sales of the shop, substitute $B = 500$, and $p = 2.00$ in the above equation.

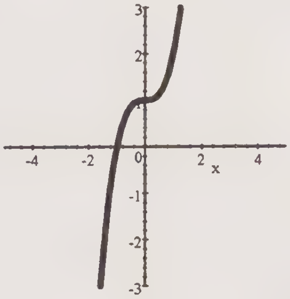
$$S = \frac{18\left(500\right)}{2} = 4500$$

The expected sales of the shop will be $4,500$ bagels per week.

PRACTICE TEST

1. Determine whether each graph is the graph of a one-to-one function.





Given $f(x) = \sqrt{x-4}$ *and* $g(x) = \sqrt{x-5}$ *find the following and determine the domain in each case.*

2. $(f+g)(x)$

3. $\left(\frac{f}{g}\right)(x)$

4. Compute the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the function $f(x) = 2x^2 - 4x + 1$.

Find the inverse of each function, and state the domain and range of g and g^{-1}.

5. $g(x) = x^2 + 2, \quad x \geq 0$

6. $g(x) = x^2 - 4x,\ x \geq 2$

Evaluate the indicated functions, where $f(x) = x - 3$, $g(x) = x^2 + 1$.

7. $(g \circ f)(0)$

8. $(f \circ g)(2)$

9. $(f + g)\left(\frac{1}{2}\right)$

10. The distance d a body falls from rest varies directly as the square of the time t of the fall. If the body falls 84 feet in 2 seconds, how far will it fall in 5 seconds?

11. Verify that $g(x) = \sqrt[3]{\frac{x-1}{2}}$ is the inverse of $f(x) = 2x^3 + 1$.

CHAPTER FIVE

Linear and Quadratic Functions

5.1 Linear Functions
5.2 Quadratic Functions

5.1 LINEAR FUNCTIONS

Definition A function of the form

$$f(x) = mx + b, \qquad m \neq 0$$

where m and b are real numbers is a **linear function** of x.

Remark: The graph of $y = mx + b$, or $f(x) = mx + b$ is a nonvertical straight line.

Example 1 Graph of a Linear Function

Graph: $f(x) = x + 2$

Solution

The equation $y = x+2$ is graphed using the intercepts, just like the way we graphed linear equations in chapter 3. The $x-intercept = (-2, 0)$, and the $y-intercept = (0, 2)$. Plot these two points, and connect them with a straight line.

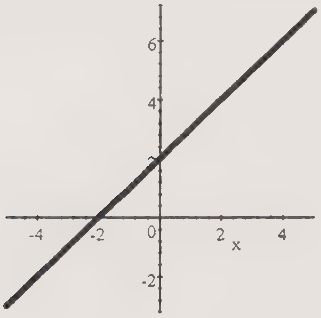


Figure 5.1

Slope of a Line

The slope m of the line containing the points (x_1, y_1) and (x_2, y_2) is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_2 \neq x_1$$

Remark: The slope of a vertical line is *undefined.*
Remark: The slope of a horizontal line is *zero.*
Remark: When determining the slope, it does not matter which point is labeled (x_1, y_1) or (x_2, y_2).

Example 2 Find the Slope of a Line

Find the slope of a line that passes through the points whose coordinates are given.

a. $(3, 1), (3, -2)$
b. $(1, 0), (-2, 3)$

Solution

a. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2-1}{3-3} = \dfrac{-3}{0} = undefined$

b. $m = \dfrac{3-0}{-2-1} = \dfrac{3}{-3} = -1$

Forms of Linear Equations

SLOPE-INTERCEPT FORM

The slope-intercept form of the equation of a line is

$$y = mx + b$$

where m is the slope and b is the $y - intercept$.

POINT-SLOPE FORM

The point-slope form of the equation of a line is

$$y - y_1 = m(x - x_1)$$

where m is the slope of the line passing through the point (x_1, y_1).

Remark: Unlike other forms of the equation of a line, this particular form is used to find the equation of the line.

GENERAL FORM

The general form of the equation of a line is

$$ax + by + c = 0$$

where a, b, and c are real numbers and a and b are not both zero.

STANDARD FORM

The standard form of the equation of a line is

$$ax + by = c$$

where a, b, and c are real numbers and a and b are not both zero.

HORIZONTAL LINE

Equation of a horizontal line is

$$y = k$$

where k is a constant.

VERTICAL LINE

Equation of a vertical line is

$$x = k$$

where k is a constant.

Equation of a Line

The Equation of a Line Given the Slope and the Y-intercept

Substitute the slope, m, and the $y - intercept$, b, in the slope-intercept formula $y = mx + b$.

Example 3 **Find the Equation of a Line**

Find the equation of the line with slope 3, and $y - intercept$ $(0,5)$.

Solution

Use the slope-intercept form with $m = 3$ and $b = 5$.

$$y = mx + b = 3x + 5$$

The Equation of a Line Given the Slope and One Point

Substitute the slope, m, and the coordinates of the given point (x_1, y_1) in the point-slope form of the equation of the line $y - y_1 = m(x - x_1)$.

Example 4 **Find the Equation of a Line**

Determine, in standard form, the equation of the line that passes through the point $(2, -1)$ with slope -2.

Solution

Use the point-slope form with $m = -2$, $x_1 = 2$, and $y_1 = -1$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-1) &= -2(x - 2) \\ y + 1 &= -2x + 4 \\ 2x + y &= 3 \end{aligned}$$

The Equation of a Line Given Two Points

Use the Coordinates of two points and find the slope of the line. Then, use the slope and either of the points in the point-slope formula to find the equation of the line.

Example 5 **Find the Equation of a Line**

Determine, in general form, the equation of the line that passes through the points $(3, 1)$ and $(0, 4)$.

Solution

Use the slope formula to find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{0 - 3} = \frac{3}{-3} = -1$$

Use the point-slope formula with $m = -1$, $x_1 = 0$, and $y_1 = 4$.

$$\begin{aligned} y - 4 &= -1(x - 0) \\ y - 4 &= -x \\ x + y - 4 &= 0 \end{aligned}$$

Parallel and Perpendicular Lines

Points to Remember

Slopes of two parallel lines are the same.

Slopes of two perpendicular lines are negative reciprocals of one another, or $m_1 \cdot m_2 = -1$.

Example 6 **Find the Equation of a Line**

Find an equation for the line with the given properties. Express your answer in standard form.

a. Pass through $(1,-2)$, and parallel to $2x+y=9$

b. Pass through $(-4,1)$, and perpendicular to $8x+4y=1$

Solution

a. Determine the slope of the line $2x+y=9$ by writing it in slope-intercept form.

$$2x+y=9$$
$$y=-2x+9$$

Since the slope of the line $2x+y=9$ is -2, the slope of the parallel line is also -2.

Now, use the point-slope formula with $m=-2$, $x_1=1$, and $y_1=-2$.

$$y-(-2)=-2\,(x-1)$$
$$y+2=-2x+2$$
$$2x+y=0$$

b. Determine the slope of the line $8x+4y=1$ by writing it in slope-intercept form.

$$8x+4y=1$$
$$4y=-8x+1$$
$$y=\tfrac{-8}{4}x+\tfrac{1}{4}$$
$$y=-2x+\tfrac{1}{4}$$

Since the slope of the line $8x+4y=1$ is -2, the slope of the perpendicular line is $\frac{1}{2}$.

Now, use the point-slope formula with $m=\frac{1}{2}$, $x_1=-4$, and $y_1=1$.

$$y-1=\tfrac{1}{2}\,[x-(-4)]$$
$$y-1=\tfrac{1}{2}\,[x+4]$$
$$y-1=\tfrac{1}{2}x+2$$
$$2y-2=x+4$$
$$-x+2y=6$$

Applications

In this section we will show how to find the equation of a linear relationship which arises in some business and economic applications.

Profit of a firm is the difference between the revenue and total cost. Profit function is defined as follows:

Profit = Total revenue −Total cost

When total revenue $>$ total cost, there is **net profit.**

When total revenue $<$ total cost, there is **net loss.**

The value of x when $R(x)=C(x)$ is called **break-even** point. At **break-even** point $P(x)=0$.

Example 7 **Find the Break-even Point**

The cost incurred in manufacturing a certain type of radio is \$120,000 plus an additional \$54 per radio.

a. Determine the profit function given that x radios are manufactured and sold at \$78.

b. Determine the break-even point.

Solution

a. The revenue function is $R(x)=78x$. The cost function is $C(x)=54x+120{,}000$. Thus, the profit function is

$$P(x) = 78x - (54x + 120,000)$$
$$P(x) = 24x - 120,000$$

b. At break-even point $P(x) = 0$.

$$0 = 24x - 120,000$$
$$-24x = -120,000$$
$$x = 5,000$$

If $5,000$ units are manufactured the firm will break even.

Example 8 **Find a Linear Model**

The book value of a machine is \$4500. It is estimated that after 9 years the value of machine will be \$900. If value of the machine is a function of the machine's age;

a. Find the linear function demonstrating this relationship.

b. Determine the value of machine after 4 years.

Solution

a. The two data points are $(0, 4500)$ and $(9, 900)$. The slope-intercept relationship will have the form

$$V(t) = mt + b$$

Use the two data points to find the slope.

$$m = \frac{V_2 - V_1}{t_2 - t_1} = \frac{4500 - 900}{0 - 9} = \frac{3600}{-9} = -400$$

Use $m = -400$, $V_1 = 900$, and $t_1 = 9$ in the point-slope formula.

$$V - V_1 = m(t - t_1)$$
$$V - 900 = -400(t - 9)$$
$$V - 900 = -400t + 3600$$
$$V = -400t + 4500$$

Note: A shortcut to finding the above function is to use $m = -400$, and $y - intercept = 4500$ in the equation $y = mx + b$.

b. To find the value of machine after 4 years, use $t = 4$ in the equation

$$V(t) = -400t + 4500$$
$$V(4) = -400(4) + 4500 = 2900$$

Therefore, the value of machine after 4 years is \$2900.

5.2 QUADRATIC FUNCTIONS

Definition A function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$ is called a **quadratic function.**

Remark: The graph of a quadratic function is a **parabola.**

Forms of Quadratic Functions

GENERAL FORM

A quadratic function written in the form

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

is in **general form.**

The graph of function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

has the following characteristics:

a. The vertex of the graph is at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ – **This is called the vertex formula**

b. The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.

c. The graph opens up when $a > 0$, and opens down when $a < 0$.

Example 9 **Find the Vertex of a Quadratic Function**

Use the vertex formula to find the vertex of the quadratic function $f(x) = -2x^2 + 4x + 3$.

Solution

$f(x) = -2x^2 + 4x + 3$

$$h = -\frac{b}{2a} = -\frac{4}{2(-2)} = 1$$

$$k = f\left(-\frac{b}{2a}\right) = f(1) = -2(-1)^2 + 4(1) + 3 = -2 + 4 + 3 = 5$$

The vertex is $(1, 5)$.

STANDARD FORM

A quadratic function written in the form

$$f(x) = a(x - h)^2 + k, \qquad a \neq 0$$

is in **standard form.**

The graph of function

$$f(x) = a(x - h)^2 + k, \qquad a \neq 0$$

has the following characteristics:

a. The vertex of the graph is at (h, k).
b. The axis of symmetry is the vertical line $x = h$.
c. The graph opens up if $a > 0$, and opens down if $a < 0$.

Remark Every quadratic function given in general form can be written in **standard form** using the method of completing the squares.

Example 10 **Find the Standard Form of a Quadratic Function**

Use the method of completing the squares to find the standard form of the quadratic function $f(x) = 4x^2 - 24x + 20$. Sketch the graph.

Solution

Step 1: Factor 4 from the variable terms.

$$f(x) = 4\left(x^2 - 6x\right) + 20$$

Step 2: Complete the square by adding and subtracting $\left(-\frac{6}{2}\right)^2$ within parentheses.

$$f(x) = 4\left(x^2 - 6x + 9 - 9\right) + 20$$

Step 3: Regroup by taking $4(-9)$ out of the parentheses.

$$f(x) = 4\left(x^2 - 6x + 9\right) + 20 - 36$$

Step 4: Factor and simplify to obtain the standard form.

$$f(x) = 4(x-3)^2 - 16$$

The vertex is $(3, -16)$. The axis of symmetry is $x = 3$. Since $a > 0$, the parabola opens up. See figure 5.1.

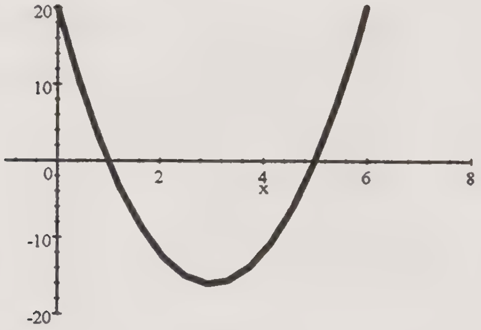


Figure 5.2

Remark: Knowledge of where the graph of f crosses the x and $y-axis$ provides additional piece of information for accurate graphing.

In order to find the intercepts, use the general formula. The $x - intercepts$ are found by solving the equation

$$ax^2 + bx + c = 0.$$

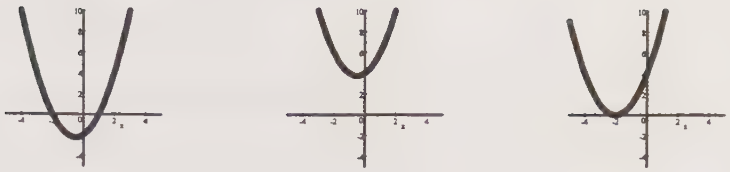
This equation has two, one or no real solutions, depending on whether the discriminant $b^2 - 4ac$ is positive, zero, or negative. Thus, the three cases are discussed as follows:

a. If $b^2 - 4ac > 0$, then the graph of f has two distinct $x - intercepts$, and it crosses the $x - axis$.

b. If $b^2 - 4ac < 0$, then the graph of f has no $x - intercepts$, and it does not cross the $x - axis$.

c. If $b^2 - 4ac = 0$, then the graph of f has one $x - intercept$, and it is tangent to the $x - axis$ at its vertex.

The Figures below illustrate these possibilities for parabolas that open up.



$b^2 - 4ac > 0$ $\qquad$ $b^2 - 4ac < 0$ $\qquad$ $b^2 - 4ac = 0$

Figure 5.3

Applications

We know that the graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola with vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. This vertex is the highest point on the graph of function f if $a < 0$. If the vertex is highest point on the graph of f then $f\left(-\frac{b}{2a}\right)$ is the **maximum value** of f. If the vertex is the lowest point on the graph $(a > 0)$ then $f\left(-\frac{b}{2a}\right)$ is the **minimum value** of f. We will use these ideas to solve some applications of quadratic functions.

Example 11 **Find the Maximum Revenue**

The manufacturer of a digital watch has found that when the watches are sold at a price of p dollars per unit, the revenue R in dollars as a function of p is

$$R(p) = -500p^2 + 15{,}000p$$

Determine the unit price which maximizes the revenue. Also, find the maximum revenue.

Solution

The function R is a quadratic function with $a = -500$ and $b = 15{,}000$. The unit price that produces maximum profit is given by

$$p = -\frac{b}{2a} = -\frac{15{,}000}{2(-500)} = \$15$$

The maximum revenue is

$$R(15) = -500(15)^2 + 15{,}000(15) = \$112{,}500$$

Example 12 **Solve a Projectile Application**

A ball is thrown vertically upward from a height of 10 feet with an initial velocity of 36 feet per second. The formula

$$s(t) = -16t^2 + 36t + 10$$

determines the height of the ball above the ground t seconds after it is thrown.

a. How long does it take the ball to reach its maximum height?
b. What is the maximum height that the ball reaches?
c. When does the ball hit the ground?

Solution

a. The graph of the function $s(t) = -16t^2 + 36t + 10$ is a parabola that opens downward. It has a maximum value at its vertex. The time taken by the ball to attain its maximum height is given by

$$t = -\frac{b}{2a} = -\frac{36}{2(-16)} = \frac{9}{8}$$

Therefore, it takes the ball 1.125 seconds to reach its maximum height.

b. The maximum height the ball reaches is given by

$$s\left(\frac{9}{8}\right) = -16\left(\frac{9}{8}\right)^2 + 36\left(\frac{9}{8}\right) + 10 = 30.25 \text{ feet}$$

Thus, the maximum height is 30.25 feet.

c. The ball will hit the ground when $s(t) = 0$. Therefore, solve
$-16t^2 + 36t + 10 = 0$

$$t = \frac{-36 \pm \sqrt{(36)^2 - 4(-16)(10)}}{2(-16)}$$

$$= \frac{-36 \pm \sqrt{1296 + 640}}{-32} = \frac{-36 \pm 44}{-32}$$

$$t = 2\tfrac{1}{2} \text{ or } -\tfrac{1}{4} \text{ seconds}$$

The negative answer is discarded, thus the ball hits the ground in $2\frac{1}{2}$ seconds.

SOLVED PROBLEMS

Slopes of Lines

Find the slope of the line that passes through the given points.

1. $(-2,3)$ and $(3,1)$

Solution

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-3}{3-(-2)}=\frac{-2}{5}$$

The slope of the line is $\frac{-2}{5}$

2. $(2,5)$ and $(0,5)$

Solution

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{5-5}{0-2}=\frac{0}{-2}=0$$

The slope of the line is zero. The line that passes through the points $(2,5)$ and $(0,5)$ is a horizontal line.

3. $(-5,2)$ and $(-5,4)$

Solution

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-2}{-5-(-5)}=\frac{2}{0}=undefined$$

The slope of the line is undefined. The line that passes through the points $(-5,2)$ and $(-5,4)$ is a vertical line.

Equations of Lines

Equation of a Line Given the Slope and the Y-intercept

Find the equation of the straight line that satisfies the given conditions. Express your answer in standard form.

4. Slope $=-3$, passing through $(-1,4)$

Solution

Use the point-slope formula with $m=-3$ and $(x_1,y_1)=(-1,4)$.

$$y-\mathbf{4}=\mathbf{-3}\,[x-(\mathbf{-1})]$$
$$y-4=-3\,(x+1)$$
$$y+3x=1$$

The equation of the line in standard form is $3x+y=1$.

5. Slope $=-\frac{1}{4}$, passing through $(8,3)$

Solution

Use the point-slope formula with $m=-\frac{1}{4}$ and $(x_1,y_1)=(8,3)$.

$$y-\mathbf{3}=-\frac{\mathbf{1}}{\mathbf{4}}\,(x-\mathbf{8})$$
$$\mathbf{4}\,(y-3)=\mathbf{4}\left[-\frac{1}{4}\,(x-8)\right]$$
$$4y-12=-x+8$$
$$4y+x=20$$

The equation of the line in standard form is $x+4y=20$.

6. Slope $= -1$, passing through $(3, 0)$

Solution

Use $m = -1$ and $(x_1, y_1) = (3, 0)$ in the point-slope formula to get

$$\begin{aligned} y - \mathbf{0} &= \mathbf{-1}\,(x - \mathbf{3}) \\ y &= -x + 3 \\ y + x &= 3 \end{aligned}$$

The equation of the line in standard form is $x + y = 3$.

7. Slope $= undefined$, passing through $(-2, 1)$

Solution

Use $m = undefined$, $x_1 = -2$, $y_1 = 1$ in the point-slope formula.

$$y - 1 = undefined\,[x - (-2)]$$
$$y - 1 = undefined\,(x + 2)$$

or

$$\frac{y-1}{x+2} = undefined$$

In order for a rational expression to be undefined, the denominator must be zero. Thus,

$$x + 2 = 0$$

The equation of the line in standard form is $x = -2$.

Equation of a Line Given Two Points

Find the equation of the line that satisfies the given conditions. Express your answer in slope-intercept form.

8. Passing through $(3, -2)$ and $(1, 4)$

Solution

Use $x_1 = 3$, $y_1 = -2$, $x_2 = 1$, and $y_2 = 4$ to find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{1 - 3} = -3$$

Use the point-slope formula with $m = -3$, $x_1 = 3$, and $y_1 = -2$.

$$\begin{aligned} y - (\mathbf{-2}) &= \mathbf{-3}\,(x - \mathbf{3}) \\ y + 2 &= -3x + 9 \\ y + 3x &= 7 \\ y &= -3x + 7 \end{aligned}$$

The equation of the line is $y = -3x + 7$.

9. Passing through $(-8, -1)$ and $(4, 3)$

Solution

Use $x_1 = -8$, $y_1 = -1$, $x_2 = 4$, and $y_2 = 3$ to find the slope

$$m = \frac{3 - (-1)}{4 - (-8)} = \frac{4}{12} = \frac{1}{3}$$

Use the point-slope formula with $m = \frac{1}{3}$, $x_1 = -8$, and $y_1 = -1$.

$$\begin{aligned} y - (\mathbf{-1}) &= \mathbf{\frac{1}{3}}\,[x - (\mathbf{-8})] \\ y + 1 &= \frac{1}{3}\,(x + 8) \\ \mathbf{3}\,[y + 1] &= \mathbf{3}\left[\frac{1}{3}\,(x + 8)\right] \\ 3y + 3 &= x + 8 \end{aligned}$$

$$3y = x + 5$$

$$y = \frac{1}{3}x + \frac{5}{3}$$

The equation of the line is $y = \frac{1}{3}x + \frac{5}{3}$.

Equation of a Line Given the Slope and an Intercept

Find the equation of the line that satisfies the given conditions. Express your answer in standard form.

10. Slope $= \frac{1}{3}, y - intercept = -2$

Solution

Use the slope- intercept formula with $m = \frac{1}{3}$ and $b = -2$.

$$y = \frac{1}{3}x - 2$$

$$3\left(y\right) = 3\left(\frac{1}{3}x - 2\right)$$

$$3y - x = -6$$

The equation of the line is $-x + 3y = -6$ or $x - 3y = 6$.

11. Slope $= -2, x - intercept = 3$

Solution

Use the point slope formula with $m = -2$, $x_1 = 3$, and $y_1 = 0$.

$$y - 0 = -2\left(x - 3\right)$$

$$y = -2x + 6$$

$$y + 2x = 6$$

The equation is $2x + y = 6$.

Equation of a Line Given the Two Intercepts

12. Find an equation, in standard form, of a line with $x - intercept = -2$ and $y - intercept = 3$.

Solution

$$x - intercept = (-2, 0)$$

$$y - intercept = (0, 3)$$

Use $x_1 = -2$, $y_1 = 0$, $x_2 = 0$, and $y_2 = 3$ to find the slope.

$$m = \frac{3 - 0}{0 - (-2)} = \frac{3}{2}$$

Use $m = \frac{3}{2}$, and $y - intercept = 3$ in the slope-intercept formula.

$$y = \frac{3}{2}x + 3$$

$$2\left(y\right) = 2\left(\frac{3}{2}x + 3\right)$$

$$2y = 3x + 6$$

$$2y - 3x = 6$$

The equation of the line in standard form is $-3x + 2y = 6$.

Parallel and Perpendicular Lines

Find an equation, in standard form, of the straight line that satisfies the given conditions.

13. Passing through $(-2, 1)$, parallel to $3x - 2y = 6$

Solution

Determine the slope of the line $3x - 2y = 6$ by writing it in slope-intercept form:

$$3x - 2y = 6$$
$$-2y = -3x + 6$$
$$y = \frac{3}{2}x - 3$$

Since the slope of line $3x - 2y = 6$ is $\frac{3}{2}$, the slope of the parallel line is also $\frac{3}{2}$.

Now, use the point-slope formula with $m = \frac{3}{2}$, $x_1 = -2$, and $y_1 = 1$ to find the desired equation.

$$\mathbf{y - 1 = \frac{3}{2}\left[x - (-2)\right]}$$
$$y - 1 = \frac{3}{2}(x + 2)$$
$$\mathbf{2}[y - 1] = \mathbf{2}\left[\frac{3}{2}(x + 2)\right]$$
$$2y - 2 = 3(x + 2)$$
$$2y - 2 = 3x + 6$$
$$2y - 3x = 8$$

The equation of the line is $-3x + 2y = 8$.

14. Passing through $(-3, 1)$, perpendicular to $4x - 3y = 1$

Solution

Determine the slope of the line $4x - 3y = 1$ by writing it in the slope-intercept form:

$$4x - 3y = 1$$
$$-3y = -4x + 1$$
$$y = \frac{4}{3}x - \frac{1}{3}$$

Since the slope of the line $4x - 3y = 1$ is $\frac{4}{3}$, the slope of the perpendicular line is $-\frac{3}{4}$.

Therefore, use $m = -\frac{3}{4}$, $x_1 = -3$, and $y_1 = 1$ in the point-slope formula.

$$\mathbf{y - 1 = -\frac{3}{4}\left[x - (-3)\right]}$$
$$y - 1 = -\frac{3}{4}[x + 3]$$
$$4y - 4 = -3[x + 3]$$

- Multiply each side of the equation by 4.

$$4y - 4 = -3x - 9$$
$$4y + 3x = -5$$

- Add 4 to each side.

The equation of the line is $3x + 4y = -5$.

15. Are the lines $4x - 2y = 3$ and $x + 2y = 5$ perpendicular?

Solution

Find the slope of each line.

$$\begin{array}{ll} 4x - 2y = 3 & x + 2y = 5 \\ -2y = -4x + 3 & 2y = -x + 5 \\ y = 2x - \frac{3}{2} & y = -\frac{1}{2}x + \frac{5}{2} \\ m_1 = 2 & m_2 = -\frac{1}{2} \end{array}$$

$$m_1 \cdot m_2 = 2\left(-\frac{1}{2}\right) = -1$$

The lines are perpendicular, as the product of their slopes is -1.

16. Find the equation of the horizontal line containing the point $(-1, 2)$.

Solution

The slope of a horizontal line is zero. Use $m = 0$, $x_1 = -1$, and $y_1 = 2$ in the point-slope formula.

$$y - \mathbf{2} = 0\left[x - (-\mathbf{1})\right]$$

$$y - 2 = 0$$

The equation of the line is $y = 2$.

17. Find the equation of the vertical line passing through the point $(2, -3)$.

Solution

The slope of a vertical line is undefined. Use $m = undefined$, $x_1 = 2$, and $y_1 = -3$ in the point-slope formula.

$$y - (-\mathbf{3}) = undefined\,(x - \mathbf{2})$$

or

$$\frac{y+3}{x-2} = undefined$$

In order for a fraction to be undefined, the denominator must be zero. Thus,

$$x - 2 = 0$$

The equation of the line is $x = 2$.

Applications

18. An application store estimated that 40 transistor radios will be sold per day during an upcoming holiday season if the new radios are priced at \$100. At a price of \$150, 25 units are expected to be sold.

a. Find a linear function that expresses the demand as a function of the price. (write equation in slope-intercept form)

b. Find the expected demand if the price of the radio is \$200.

Solution

a. The two data points have coordinates $(100, 40)$ and $(150, 25)$. The slope-intercept relationship will have the form

$$d(p) = mp + k$$

Given the two data points, the slope is

$$m = \frac{d_2 - d_1}{p_2 - p_1} = \frac{40 - 25}{100 - 150} = \frac{15}{-50} = -0.30$$

Use $m = -0.30$, $d_1 = 25$, and $p_1 = 150$ in the point-slope equation.

$$d - d_1 = m(p - p_1)$$

$$d - 25 = -0.30(p - 150)$$

$$d - 25 = -0.30p + 45$$

$$d = -0.30p + 70$$

b. Having found the demand function, substitute the value of p into the equation $d = -0.30p + 70$. Thus,

$$d(200) = -0.30(200) + 70$$
$$d = 10$$

The expected demand for the transistor radios is 10, if the price is \$200.

19. A police department believes that the number of serious crimes which occur each month is dependent upon the number of police officers for preventive purposes. If 140 police officers are assigned for preventive patrol, there will be no crime. However, it is expected that assignment of no police officers would result in 1500 serious crimes per month. Find a linear function demonstrating this relationship.

Solution

The two data points are $(140, 0)$ and $(0, 1500)$. First find the slope of the line joining the two points.

$$m = \frac{c_2 - c_1}{p_2 - p_1} = \frac{1500 - 0}{0 - 140} = -10.7$$

The slope-intercept relationship will have the form

$$c(p) = mp + k$$

Use $m = -10.7$, $c_1 = 0$, and $p_1 = 140$ in the equation

$$c - c_1 = m(p - p_1)$$
$$c - 0 = -10.7(p - 140)$$
$$c = -10.7p + 1498$$

20. A doll company had a profit of \$50,000 per year when it had sold 1,600 dolls. When the sales of dolls increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P, is a linear function of the number of dolls sold.

a. Find the profit function.

b. How many dolls must be sold to break even?

Solution

a. The two data points are $(1600, \$50000)$ and $(1800, \$60000)$. The slope-intercept relationship will have the form

$$P(x) = mx + k$$

Given the data points, the slope is

$$m = \frac{P_2 - P_1}{x_2 - x_1} = \frac{60,000 - 50,000}{1,800 - 1,600} = 50$$

Use the point-slope formula with $m = 50$, $P_1 = 50000$, and $x_1 = 1600$.

$$P - P_1 = m(x - x_1)$$
$$P - 50,000 = 50(x - 1600)$$
$$P - 50,000 = 50x - 80,000$$
$$P = 50x - 30,000$$

b. We can find the number of dolls that must be sold to break even by setting the profit function equal to zero and solving for x.

$$P(x) = 50x - 30,000$$
$$0 = 50x - 30,000$$
$$-50x = -30,000$$
$$x = 600$$

If the company sells 600 dolls it will break even.

Quadratic Functions

Use the method of completing the squares to write the quadratic functions in standard form.

21. $f(x) = -2x^2 + 12x - 13$

Solution

$$\begin{aligned} f(x) &= -2x^2 + 12x - 13 && \\ &= -2\left(x^2 - 6x\right) - 13 && \bullet \text{ Factor out 2 from the x-terms.} \\ &= -2\left(x^2 - 6x + \mathbf{9} - \mathbf{9}\right) - 13 && \bullet \text{ Complete the square.} \\ &= -2\left(x^2 - 6x + 9\right) - 2(-9) - 13 && \bullet \text{ Regroup.} \\ &= -2(x-3)^2 + 18 - 13 && \bullet \text{ Factor and simplify.} \\ f(x) &= -2(x-3)^2 + 5 && \bullet \text{ Standard form.} \end{aligned}$$

22. $g(x) = x^2 + 4x - 2$

Solution

$$\begin{aligned} g(x) &= x^2 + 4x - 2 && \\ &= \left(x^2 + 4x + \mathbf{4} - \mathbf{4}\right) - 2 && \bullet \text{ Complete the square.} \\ &= \left(x^2 + 4x + 4\right) - 4 - 2 && \bullet \text{ Regroup.} \\ g(x) &= (x+2)^2 - 6 && \bullet \text{ Standard form.} \end{aligned}$$

23. $f(x) = -x^2 + x + \frac{7}{4}$

Solution

$$\begin{aligned} f(x) &= -x^2 + x + \frac{7}{4} && \\ &= -\left(x^2 - x\right) + \frac{7}{4} && \\ &= -\left(x^2 - x + \mathbf{\frac{1}{4}} - \mathbf{\frac{1}{4}}\right) + \frac{7}{4} && \bullet \text{ Complete the square.} \\ &= -\left(x^2 - x + \frac{1}{4}\right) - \left(-\frac{1}{4}\right) + \frac{7}{4} && \bullet \text{ Regroup.} \\ &= -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} + \frac{7}{4} && \\ f(x) &= -\left(x - \frac{1}{2}\right)^2 + 2 && \bullet \text{ Factor and simplify.} \end{aligned}$$

Use the vertex formula to determine the vertex of the graph of the function and write the function in standard form.

24. $f(x) = x^2 + 5x + 1$

Solution

$$f(x) = x^2 + 5x + 1 \qquad \bullet\ a = 1, b = 5, c = 1.$$

$$h = \frac{-b}{2a} = \frac{-5}{2}$$

$$\begin{aligned} k = f\left(\frac{-b}{2a}\right) &= \left(\frac{-5}{2}\right)^2 + 5\left(\frac{-5}{2}\right) + 1 \\ &= \frac{25}{4} - \frac{25}{2} + 1 = -\frac{21}{4} \end{aligned}$$

The vertex is $\left(\frac{-5}{2}, \frac{-21}{4}\right)$. Substitute the values of h and k into the standard form

$$f(x) = a(x-h)^2 + k$$

to get

$$f(x) = \left[x - \left(-\frac{5}{2}\right)\right]^2 + \left(-\frac{21}{4}\right)$$

$$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{21}{4}$$

25. $f(x) = -x^2 + 6x + 4$

Solution

$$f(x) = -x^2 + 6x + 4$$

• $a = -1, b = 6, c = 4.$

$$h = \frac{-b}{2a} = \frac{-6}{2(-1)} = 3$$

$$k = f\left(\frac{-b}{2a}\right) = f(3) = -(3)^2 + 6(3) + 4$$
$$= -9 + 18 + 4 = 13$$

The vertex is $(3, 13)$. Substitute the values of h and k into the standard form

$$f(x) = a(x-h)^2 + k$$

to get

$$f(x) = -1(x-3)^2 + 13$$

Find the maximum or minimum value of the given function. State whether this value is a maximum or a minimum.

26. $f(x) = x^2 + 6x$

Solution

Since the function f is a quadratic function, the vertex formula can be used to determine the maximum or the minimum value. Since $a = 1 > 0$, the parabola opens up and the quadratic function f has a minimum.

The minimum value of quadratic function is the $y-coordinate$ of the vertex.

$$h = \frac{-b}{2a} = \frac{-6}{2} = -3$$

$$k = f\left(\frac{-b}{2a}\right) = f(-3) = (-3)^2 + 6(-3) = -9$$

The $y-coordinate$ of the vertex is -9. Thus, the minimum value is -9.

27. $f(x) = 2x^2 + 5x + 1$

Solution

Since $a = 2 > 0$ the parabola opens up. Therefore, the function f has a minimum.

$$h = \frac{-b}{2a} = \frac{-5}{4}$$

$$k = f\left(\frac{-5}{4}\right) = 2\left(\frac{-5}{4}\right)^2 + 5\left(\frac{-5}{4}\right) + 1$$
$$= 2\left(\frac{25}{16}\right) - \frac{25}{4} + 1$$
$$= \frac{25}{8} - \frac{25}{4} + 1$$
$$= \frac{-17}{8}$$

The $y-coordinate$ of the vertex is $\frac{-17}{8}$. Thus, the minimum value is $-\frac{17}{8}$.

28. $f(x) = -3x^2 - 8$

Solution

Since $a = -3 < 0$, the parabola opens down. Therefore, the function f has a maximum.

$$h = \frac{-b}{2a} = \frac{0}{2(-3)} = 0$$

$$k = f(0) = -8$$

The $y-coordinate$ of the vertex is -8. Thus, the maximum value is -8.

Applications

29. Robin wants to enclose a rectangular play area for her child. She needs to fence in only three sides, as her house serves as one of the lengths of the rectangular play area. If she uses 100 feet of fencing, what dimensions of the play area would give a maximum area? (See Figure 5.4.)

Solution

From the sketch you can see that the 100 feet will be used for three sides, two of which is of the same length, w.

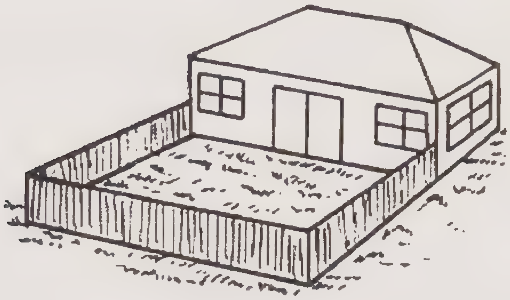


Fig. 5.4

Let l = length of the play area

w = width of the play area

Since the amount of fencing to be used is 100 feet, thus

$$2w + l = 100$$
$$l = 100 - 2w$$

The area of the enclosed region is given by

$$A = (100 - 2w)\,w$$
$$A = 100w - 2w^2$$

or

$$A(w) = -2w^2 + 100w$$

Graph of function A is a parabola that opens down and has a maximum value at its vertex (w, A).

Use the vertex formula with $a = -2$, and $b = 100$.

$$w = -\frac{b}{2a} = -\frac{100}{2(-2)} = 25$$

$$A(25) = -2(25)^2 + 100(25) = 1250$$

Therefore, the maximum area of 1250 square feet is obtained when the dimensions of the play area are:

$$w = 25$$

and

$$l = 100 - 2w = 100 - 2(25) = 50$$

30. The perimeter of a rectangular area is 220 meters. If l is the length of one side, find the maximum area that can be enclosed.

Solution

Since the perimeter of the rectangular area is 220 meters, we can express w, the width of the rectangle, in terms of l as

$$2l + 2w = 220$$
$$l + w = 110$$

$$w = 110 - l$$

Area of the enclosed region is given by

$$A = lw$$
$$A = l(110 - l)$$
$$A = 110l - l^2$$
$$A(l) = -l^2 + 110l$$

But this is the equation of a parabola. Since $a = -1 < 0$, the parabola opens downward, and the second coordinate of the vertex represents the maximum area.

Use vertex formula with $a = -1$, $b = 110$.

$$l = -\frac{b}{2a} = \frac{-110}{2(-1)} = 55$$

$$A(55) = -(55)^2 + 110(55) = 3025$$

Therefore, the maximum area that can be enclosed is $3,025$ square meters.

PRACTICE TEST

1. Find the equation of the line that passes through the point $(3, 5)$, and is parallel to the line $x = -2$.

2. Determine whether the lines $3y + 2x = 1$ and $2y - 3x = 4$ are perpendicular?

3. Determine, in slope-intercept form, the equation of the line with $x - intercept = 1$ and $y - intercept = 4$.

4. Find an equation, in standard form, of the straight line that passes through the point $(2, -1)$ and is perpendicular to the graph of $x - 3y + 6 = 0$

5. Find the maximum or minimum value of the quadratic function

$$f(x) = -2x^2 + 12x - 13$$

State whether this is a maximum or a minimum.

6. Use the method of completing the squares to write the following quadratic function in standard form.

$$f(x) = -3x^2 + 6x - 15$$

7. A farmer uses 750 feet of fencing to enclose a rectangular region and also to divide it into two regions with a fence parallel to the shortest side. Find the largest area that can be enclosed.

8. Determine, in standard form, the equation of the line that passes through the points $(-1, 3)$ and $(1, 2)$.

9. A firm sells a product for $\$16.50$ per unit. Variable costs per unit are $\$4.50$, and fixed costs equal $\$1,800$ per month. How many units must be sold in order to break even?

10. A company purchased a machine for \$8000 in 1993. The price of the machine declines to \$2000 after 6 years. Write a linear equation that gives the value of machine, V, in terms of t years.

CHAPTER SIX

Polynomial and Rational Functions

6.1 Polynomial Division
6.2 Graphs of Polynomial Functions
6.3 Zeros of Polynomial Functions
6.4 Complex Polynomial Functions
6.5 Rational Functions and their Graphs

6.1 POLYNOMIAL DIVISION

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

or

$$\underbrace{f(x)}_{dividend} = \underbrace{g(x)}_{divisor} \cdot \underbrace{q(x)}_{quotient} + \underbrace{r(x)}_{remainder}$$

where the degree of $r(x)$ is less than the degree of $g(x)$.

Example 1 **Divide Polynomials**

Perform the indicated operation: $(x^3 - 5x^2 + 11x - 10) \div (x - 2)$

Solution

$$\begin{array}{r}
x^2 - 3x + 5 \\
x - 2\overline{)x^3 - 5x^2 + 11x - 10} \\
\underline{x^3 - 2x^2} \qquad\qquad\quad \\
-3x^2 + 11x \qquad\; \\
\underline{-3x^2 + 6x} \qquad\; \\
+5x - 10 \\
\underline{+5x - 10} \\
0 \quad
\end{array}$$

$$\left(x^3 - 5x^2 + 11x - 10\right) \div (x - 2) = x^2 - 3x + 5$$

Synthetic Division

Synthetic division is a shortened version of long division. In performing synthetic division the following steps are used:

1. Arrange the dividend with variables in descending powers of x, and write any missing term (including constant) with a zero coefficient.
2. Write the divisor in the form $x - b$.
3. Copy coefficients of the dividend, and place b to the left of division symbol.
4. Bring down the leading coefficient of the dividend to row 3.
5. Multiply the first entry in row 3 by b, and place the result in row 2 in the next column.
6. Add the entry in row 2 to entry in row 1, and put the result in row 3.
7. Repeat steps 5 and 6 until all columns are filled in.
8. The final entry in row 3, is the remainder. The other entries are the coefficients of the quotient (in descending order).

Example 2 **Use Synthetic Division to Divide Polynomials**

Use synthetic division to divide polynomials.
$(2 - 7x^2 + 3x^3 + 4x^4) \div (x + 2)$

Solution

Step 1: Arrange the coefficients of the dividend in descending powers of x.

$$4 \quad 3 \quad -7 \quad 0 \quad 2$$

Step 2: Divisor is $x-(-2)$.

Step 3: Copy coefficients of the dividend, place -2 to the left of the division symbol.

$$\begin{array}{r|rrrrrl} -2 & 4 & 3 & -7 & 0 & 2 & \text{row 1} \end{array}$$

Step 4: Bring down 4 to row 3.

$$\begin{array}{r|rrrrrl} -2 & 4 & 3 & -7 & 0 & 2 & \text{row 1} \\ & & & & & & \text{row 2} \\ \hline & 4 & & & & & \text{row 3} \end{array}$$

Step 5: Multiply 4 by -2 and place the result in row 2 in the next column (col. 2).

$$\begin{array}{r|rrrrrl} -2 & 4 & 3 & -7 & 0 & 2 & \text{row 1} \\ & & -8 & & & & \text{row 2} \\ \hline & 4 & & & & & \text{row 3} \end{array}$$

Step 6: Add 3 to -8 and put the result in row 3.

$$\begin{array}{r|rrrrrl} -2 & 4 & 3 & -7 & 0 & 2 & \text{row 1} \\ & & -8 & & & & \text{row 2} \\ \hline & 4 & -5 & & & & \text{row 3} \end{array}$$

Step 7: Repeat steps 5 and 6 until all columns are filled in.

$$\begin{array}{r|rrrrrl} -2 & 4 & 3 & -7 & 0 & 2 & \text{row 1} \\ & & -8 & 10 & -6 & 12 & \text{row 2} \\ \hline & 4 & -5 & 3 & -6 & 14 & \text{row 3} \end{array}$$

Step 8: The answer is $4x^3-5x^2+3x-6+\dfrac{14}{x+2}$

The Remainder Theorem

If a polynomial function $f(x)$ is divided by $x-c$, the remainder is $f(c)$.

Example 3 **Find the Remainder of a Polynomial Using the Remainder Theorem**

Find the remainder if $f(x)=x^2+5x-2$ is divided by $x-3$.

Solution

By the remainder theorem, the remainder is

$$r=f(3)=(3)^2+5(3)-2=22$$

Example 4 **Find the Remainder of a Polynomial Using Synthetic Division**

Given $f(x)=8x^2+15x-7$, find $f(1)$ using the synthetic division.

Solution

Step 1: Arrange the coefficients of the dividend in descending powers of x.

$$8 \quad 15 \quad -7$$

Steps 2&3: Rewrite coefficients of the dividend, place 1 to the left of the division symbol.

$$\begin{array}{r|rrrl} 1 & 8 & 15 & -7 & \text{row 1} \end{array}$$

Step 4: Bring down 8 to row 3.

$$\begin{array}{lrrrl} 1) & \overline{8} & \overline{15} & \overline{-7} & \text{row 1}\\ & & & & \text{row 2}\\ \cline{2-4} & 8 & & & \text{row 3}\end{array}$$

Step 5: Multiply 8 by 1 and place the result in row 2 in the next column.

$$\begin{array}{lrrrl} 1) & \overline{8} & \overline{15} & \overline{-7} & \text{row 1}\\ & & 8 & & \text{row 2}\\ \cline{2-4} & 8 \nearrow & & & \text{row3}\end{array}$$

Step 6: Add 15 to 8 and place the result in row 3.

$$\begin{array}{lrrrl} 1) & \overline{8} & \overline{15} & \overline{-7} & \text{row 1}\\ & & 8 & & \text{row 2}\\ \cline{2-4} & 8 & 23 & & \text{row 3}\end{array}$$

Step 7: Multiply +23 by +1 and place the result in row 2 in the next column (col. 3).

$$\begin{array}{lrrr} 1) & \overline{8} & \overline{15} & \overline{-7}\\ & & 8 & 23\\ \cline{2-4} & 8 & 23 & 16\end{array}$$

Step 8: $f(1) = 16$

The Factor Theorem

A polynomial function $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$.

Example 5 **Find a Factor of a Polynomial**

Determine whether $x - 1$ is a factor of $f(x) = x^3 - 3x^2 + x + 1$.

Solution

$$\begin{aligned} f(1) &= (1)^3 - 3(1)^2 + (1) + 1\\ &= 1 - 3 + 1 + 1 = 0 \end{aligned}$$

Since $f(1) = 0$, it can be concluded that $x - 1$ is a factor.

6.2 GRAPHS OF POLYNOMIAL FUNCTIONS

Definition If f is a polynomial function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of polynomial function f.

Definition If r, the zero of a polynomial function repeats k times, then r is called **zero of multiplicity k.**

Example 6 **State the Multiplicity of each Zero**

Find the zeros of the polynomial function f and state the multiplicity of each zero.

$$f(x) = x(x-3)^2$$

Solution

The polynomial function f has

- 0 as a zero of multiplicity 1
- 3 as a zero of multiplicity 2

Note:

- If r is a zero of odd multiplicity, then graph of f crosses the $x - axis$ at r.
- If r is a zero of even multiplicity, then graph of f touches the $x - axis$ at r.

Definition The degree of a polynomial function is the exponent of the variable in the leading term. Constant, linear, and quadratic functions are polynomial functions of degree 0, 1, and 2 respectively.

Definition A polynomial function of degree n has at most $n-1$ **turning points**.

Procedures for Graphing

To sketch the graph of a polynomial function of degree $n \geq 3$, follow these steps:

Step 1 Find the $y - intercept$, by evaluating $f(0)$. Find the $x - intercepts$, by solving $f(x) = 0$. The *x-intercepts,* are the real zeros of the function f.

Step 2 Determine whether the graph of f touches or crosses the $x - axis$ at each $x - intercept$.

Step 3 Find the maximum number of turning points.

Step 4 Determine the intervals into which the $x - intercepts$ divide the $x - axis$.

Step 5 Choose a value (x) from each interval and find the corresponding value of $f(x)$.

Step 6 Plot the points found in steps 1 and 5 and connect them to form a smooth curve.

Example 7 **Graph a Polynomial Function**

Graph: $f(x) = (x-1)^2(x+2)$

Solution

Step 1: **a.** The $y - intercept$ is $f(0) = 2$.

b. The $x - intercepts$ satisfy the equation

$$f(x) = (x-1)^2(x+2) = 0$$

Thus,

$$(x-1)^2 = 0 \quad or \quad (x+2) = 0$$

$$x = 1 \qquad\qquad x = -2$$

The $x - intercepts$ are 1 and -2.

Step 2: The $x - intercept$, 1, is a zero of even multiplicity, so the graph of f touches the $x - axis$ at 1.

The $x - intercept$, -2, is a zero of odd multiplicity, so the graph of f crosses the $x - axis$ at -2.

Step 3: The maximum number of turning points is 2.

Steps 4&5:

	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
x	-3	-1	2
f	-16	4	4

Step 6:

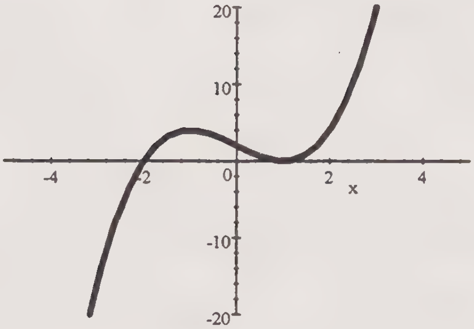


Figure 6.1

Example 8 **Graph a Polynomial Function**

Graph: $f(x) = x^3 + x^2 - 12x$
Solution
$f(x) = x^3 + x^2 - 12x$
$f(x) = x(x+4)(x-3)$
Step 1: a. The $y - intercept$ is $f(0) = 0$.
b. The $x - intercepts$ satisfy the equation

$$f(x) = x(x+4)(x-3)$$

Thus,

$$x = 0 \quad or \quad x + 4 = 0 \quad or \quad x - 3 = 0$$
$$x = 0 \qquad x = -4 \qquad x = 3$$

The $x - intercepts$ are $0, -4$ and 3.

Step 2: The $x - intercepts$, $1, -4, 3$ are zeros of odd multiplicity, so the graph of f crosses the $x - axis$ at $0, -4$, and 3.

Step 3: The maximum number of turning points is 2.

Steps 4&5:

	$-\infty < x < -4$	$-4 < x < 0$	$0 < x < 3$	$3 < x < \infty$
x	-5	-1	2	5
f	-40	12	-12	90

Step 6:

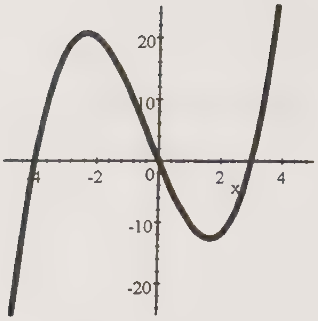


Figure 6.2

6.3 ZEROS OF POLYNOMIAL FUNCTIONS

The real zeros of a polynomial function f are real solutions of the equation $f(x) = 0$. In this section we learn some ways of finding information about the character of the zeros, which will help us find them.

Remark: A polynomial function f of degree n has at most n zeros.

The Rational Zero Theorem

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an n^{th} degree polynomial with integer coefficients. If $\frac{p}{q}$, in lowest terms, is a zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

Example 9 **Apply the Rational Zero Theorem**

Given $f(x) = 3x^3 - 13x^2 + 13x - 3$, list all possible rational zeros of f.

Solution:

List all integers p that are factors of -3 and all integers q that are factors of 3 :

$$p: \quad \pm 1, \pm 3$$
$$q: \quad \pm 1, \pm 3$$

Form all possible rational numbers using factors of p and q :

$$\frac{p}{q}: \quad \frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 3}, \frac{\pm 3}{\pm 1}$$

which simplifies to

$$\frac{p}{q}: \quad \pm 1, \pm \frac{1}{3}, \pm 3$$

Thus, the possible rational zeros of f are ± 1, $\pm \frac{1}{3}$, ± 3.

Descartes' Rule of Signs

If $f(x)$ is a polynomial with real coefficients and the terms are arranged in descending powers of x, then,

- the number of **positive real zeros** of $f(x)$ is equal to the number of variations in sign of $f(x)$ or is less than that number by an even integer.
- the number of **negative real zeros** of $f(x)$ is equal to the number of variations in sign of $f(-x)$ or is less than that number by an even integer.

Example 10 **Apply Descartes' Rule of Signs**

Determine the number of possible positive and the number of negative real zeros of the polynomial

$$f(x) = x^4 - 3x^3 + 2x^2 + 3x - 5$$

Solution

$$f(x) = x^4 - 3x^3 + 2x^2 + 3x - 5$$

There are three variations in sign of $f(x)$. By Descartes' Rule of Signs there are either three or one positive real zeros.

$$f(-x) = (-x)^4 - 3(-x)^3 + 2(-x)^2 + 3(-x) - 5$$
$$f(-x) = x^4 + 3x^3 + 2x^2 - 3x - 5$$

There is one variation in sign of $f(-x)$. By Descartes' Rule of Signs there is one negative real zero.

PROCEDURES FOR FINDING REAL ZEROS OF POLYNOMIAL FUNCTIONS

Procedures To Remember

Step 1 Use the degree of the polynomial to determine the maximum number of zeros.

Step 2 Use the Rational Zero Theorem to list all possible zeros.

Step 3 Use Descartes' Rule of Signs to determine the possible number of positive and negative zeros.

Step 4 Use synthetic division to test all possible zeros.

Step 5 Each time a zero is found, remember to use factoring methods or the quadratic formula on the depressed equation, or if necessary repeat steps 2-4 until all zeros are found.

Example 11 **Find the Zeros of a Polynomial**

Given $f(x) = 3x^3 - 13x^2 + 13x - 3$, find all rational zeros of the function f.

Solution:

From previous example all possible rational zeros of f are $\pm 1,\ \pm\frac{1}{3},\ \pm 3$.

From Descartes' Rule of Signs, the polynomial

$$f(x) = 3x^3 - 13x^2 + 13x - 3$$

has three variations in signs. Thus, there are either 1 or 3 positive real zeros. The polynomial

$$f(-x) = 3(-x)^3 - 13(-x)^2 + 13(-x) - 3$$
$$f(-x) = -3x^3 - 13x^2 - 13x - 3$$

has no variation in signs. Thus, there are no negative real zeros. Hence, the possible negative rational zeros $-1, -\frac{1}{3}, -3$ are discarded.

Now use synthetic division to test only the possible positive real zeros $1, \frac{1}{3}, 3$.

Test $x = 1$:

$$\begin{array}{r|rrrr} 1) & 3 & -13 & 13 & -3 \\ & & 3 & -10 & 3 \\ \hline & 3 & -10 & 3 & 0 \end{array}$$

Since the remainder is zero, 1 is a rational zero of f, and hence $x - 1$ is a factor of $f(x)$.

$$f(x) = (x-1)\left(3x^2 - 10x + 3\right)$$

We call the equation $3x^2 - 10x + 3 = 0$, a **depressed equation** of f.

$$f(x) = (x-1)(3x-1)(x-3)$$
$$f(x) = 3(x-1)\left(x - \frac{1}{3}\right)(x-3)$$

Therefore, the zeros of f are $1, 3, \frac{1}{3}$.

Upper and Lower Bound

Any real number that is greater than or equal to every real zero of f is called an **upper bound** for the zeros of the polynomial function. Similarly, any real number that is less than or equal to every real zero of f is called a **lower bound** for the zeros of the polynomial function.

That is,

$$l = lower\ bound \leq any\ zero\ of\ f \leq upper\ bound = u$$

UPPER AND LOWER BOUND RULE

Consider the polynomial function $f(x)$ where coefficients are real and the leading coefficient is positive. Suppose $f(x)$ is divided by $x - r$, using synthetic division:

- If $r > 0$, and all the numbers in the third row of the synthetic division table are positive or zero, then r is an **upper bound** for the real zeros of f.
- If $r < 0$, and all the numbers in the third row of the synthetic division table alternate in sign, then r is a **lower bound** for the real zeros of f.

Remark: The number zero in the third row of the synthetic division can be considered as positive or negative.

Example 12 **Find the Upper and Lower Bounds**

Find the smallest positive integer as an upper bound and the largest negative integer as a lower bound for the real zeros of $f(x) = 3x^3 - 13x^2 + 13x - 3$.

Solution:

a. Test $1, 2, 3, 4, \ldots$ as possible upper bounds by using synthetic division.

$$\begin{array}{r|rrrr} 1) & 3 & -13 & 13 & -3 \\ & & 3 & -10 & 3 \\ \hline & 3 & -10 & 3 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2) & 3 & -13 & 13 & -3 \\ & & +6 & -14 & -2 \\ \hline & 3 & -7 & -1 & -5 \end{array}$$

$$\begin{array}{r|rrrr} 3) & 3 & -13 & +13 & -3 \\ & & 9 & -12 & +3 \\ \hline & 3 & -4 & 1 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 4) & 3 & -13 & 13 & -3 \\ & & 12 & -4 & 36 \\ \hline & 3 & -1 & 9 & 33 \end{array}$$

$$\begin{array}{r|rrrr} 5) & 3 & -13 & 13 & -3 \\ & & 15 & 10 & 115 \\ \hline & 3 & 2 & 23 & 112 \end{array} \quad \rightarrow \textit{all positive signs}$$

Since the synthetic divisor $r = 5 > 0$ and all numbers in the third row of the synthetic division are positive, it can be concluded that 5 is an upper bound for the zeros of f.

b. Test $-1, -2, -3, \ldots$ as possible lower bounds by using synthetic division.

$$\begin{array}{r|rrrr} -1) & 3 & -13 & 13 & -3 \\ & & -3 & 16 & -29 \\ \hline & 3 & -16 & 29 & -32 \end{array} \quad \rightarrow \textit{alternating signs}$$

Since the synthetic divisor $r = -1 < 0$ and all numbers in the third row of the synthetic division alternate in sign, it is concluded that -1 is a lower bound for the zeros of f.

Thus,

$$-1 \le zeros\ of\ f \le 5.$$

6.4 COMPLEX POLYNOMIAL FUNCTIONS

Definition A **complex polynomial function** f of degree n is a complex polynomial function of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, and n is a nonnegative integer.

Fundamental Theorem of Algebra

Every complex polynomial function f of degree $n \geq 1$ has at least one zero in the complex number system. Real numbers are a subset of the complex number system. Thus, a polynomial with real coefficient satisfies the Fundamental Theorem of Algebra.

Extension of the Fundamental Theorem of Algebra

Every complex polynomial function $f(z)$ of degree $n \geq 1$ can be expressed as the product of n linear factors

$$f(z) = a_n (z - r_1)(z - r_2) \cdots (z - r_n)$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers.

Example 13 **Factor a Polynomial into Linear Factors**

Write $f(x) = x^3 - 4x^2 + x - 4$ as a product of linear factors.

Solution

Factoring by grouping yields

$$\begin{aligned} f(x) &= x^3 - 4x^2 + x - 4 \\ &= x^2(x-4) + (x-4) \\ &= (x-4)\left(x^2+1\right) \\ f(x) &= (x-4)(x-i)(x+i) \end{aligned}$$

Example 14 **Factor a Polynomial into Linear Factors Given One of its Zeros**

Write $f(x) = x^4 - 6x^3 + 17x^2 - 28x + 20$ as a product of linear factors, given that 2 is a zero of multiplicity 2.

Solution

The polynomial function $f(x)$ can be written as a product of linear factors. Since 2 is zero of multiplicity 2, therefore $(x-2)^2$ is a factor of the polynomial $f(x)$.

So $f(x)$ in reduced form is

$$f(x) = (x-2)^2 g(x)$$

Now $g(x)$ can be found using synthetic division twice. Thus,

2)	1	− 6	17	− 28	20
		2	− 8	18	− 20
2)	1	− 4	9	− 10	0
		2	− 4	10	
	1	− 2	5	0	

Thus,

$$g(x) = x^2 - 2x + 5$$

or

$$f(x) = (x-2)^2\left(x^2 - 2x + 5\right)$$

The zeros of the quadratic function $g(x) = x^2 - 2x + 5$ can be found using the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

By the factor theorem, polynomial $f(x)$ can be written as product of four factors.

$f(x) = (x-2)(x-2)[x-(1+2i)][x-(1-2i)]$.

Example 15 **Factor a Polynomial into Linear Factors Given One of its Zeros**

Write $f(x) = x^3 - 6x^2 + 3x + 10$ as a product of linear factors, given that 2 is a zero of $f(x)$.

Solution

The polynomial function $f(x)$ can be written as a product of linear factors. Since 2 is zero, therefore $(x-2)$ is a factor of the polynomial $f(x)$.

So $f(x)$ in reduced form is

$f(x) = (x-2)\,g(x)$

Now $g(x)$ can be found using synthetic division. Thus,

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 3 & 10 \\ & & 2 & -8 & -10 \\ \hline & 1 & -4 & -5 & 0 \end{array}$$

Thus,

$g(x) = x^2 - 4x - 5$

or $f(x) = (x-2)(x^2-4x-5) = (x-2)(x-5)(x+1)$

The Conjugate Pair Theorem

If $f(z)$ is a complex polynomial with real coefficients and $a+bi$ is a zero of f, then its conjugate $a-bi$ is also a zero of f.

Example 16 **Find a Polynomial**

Find each polynomial.

a. A polynomial of degree 3 that has zeros $4, 3, -2$.

b. A polynomial of degree 3 that has zeros $3, 2i$.

Solution

a. Because $4, 3$, and -2 are zeros, $(x-4)$, $(x-3)$, and $(x+2)$ are factors of the polynomial. Multiplying these factors will yield the indicated polynomial.

$f(x) = (x-4)(x-3)(x+2) = x^3 - 5x^2 - 2x + 24$

b. By the Conjugate Pair Theorem the polynomial also has $-2i$ as a zero. Therefore, the product of the factors $(x-3)$, $(x-2i)$, and $(x+2i)$ will produce the required polynomial.

$f(x) = (x-3)(x-2i)(x+2i) = x^3 - 3x^2 + 4x - 12$

Example 17 **Apply the Conjugate Pair Theorem to Find Zeros**

Find all zeros of $f(x) = x^4 - 6x^3 + 17x^2 - 28x + 20$, given that $1+2i$ is a zero of f.

Solution:

By the Conjugate Pair Theorem, since the coefficients of f are all real and $1+2i$ is a zero of f, then $1-2i$ is also a zero of f. Use synthetic division with $1+2i$ and then the result of this division with $1-2i$.

$$\begin{array}{r|rrrrr}
1+2i) & 1 & -6 & 17 & -28 & 20 \\
 & & 1+2i & -9-8i & 24+8i & -20 \\
\hline
1-2i) & 1 & -5+2i & 8-8i & -4+8i & 0 \\
 & & 1-2i & -4+8i & 4-8i & \\
\hline
 & 1 & -4 & 4 & 0 &
\end{array}$$

Thus, the reduced polynomial is

$$f(x) = [x-(1+2i)]\,[x-(1-2i)]\,(x^2-4x+4)$$

or

$$f(x) = [x-(1+2i)]\,[x-(1-2i)]\,(x-2)^2$$

Zeros of f are $1+2i, 1-2i$, and 2 with multiplicity of 2.

6.5 RATIONAL FUNCTIONS AND THEIR GRAPHS

A function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions, $q(x) \neq 0$ is called a **rational function.**

Asymptotes

Definition For the rational function $f(x)$ in lowest terms, and for real numbers a and b

- if $|f(x)| \to \infty$ as $x \to a$, then the line $x = a$ is a **vertical asymptote** of the graph of f.
- if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$, then the line $y = b$ is a **horizontal asymptote** of the graph of f.
- if an asymptote is neither vertical nor horizontal it is called an **oblique** (slant) **asymptote.**

DETERMINING ASYMPTOTES

Asymptotes of a rational function written in lowest terms are found using the following procedures:

Vertical Asymptotes

The rational function $f(x) = \frac{p(x)}{q(x)}$ in lowest terms will have a **vertical asymptote** $x = a$, if a is a zero of the denominator $q(x)$.

Example 18 **Find the Vertical Asymptote of a Rational Function**

Find the vertical asymptote of each rational function.

a. $f(x) = \dfrac{x^2}{x^2+2}$ **b.** $g(x) = \dfrac{2x-3}{x^2-5x+6}$

Solution

a. To find the vertical asymptotes, set the denominator equal to zero. The denominator in this case has no real zeros, therefore, the graph of f has no vertical asymptotes.

b. The zeros of the denominator are 3 and 2. Hence, the lines $x = 3$ and $x = 2$ are the vertical asymptotes of the graph of g.

Horizontal Asymptote

Let $$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} \cdots + b_1 x + b_0}$$

be a rational function with numerator of degree n and denominator of degree m.

- If $n < m$, then the line $y = 0$ is the horizontal asymptote of the graph of f.
- If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote.
- If $n > m$, then there is no horizontal asymptote.

Example 19 **Find the Horizontal Asymptote of a Rational Function**

Find the horizontal asymptote of each rational function.

a. $f(x) = \dfrac{x-3}{x^2-1}$ **b.** $g(x) = \dfrac{x^2-3}{x^2+4}$ **c.** $h(x) = \dfrac{x^2-1}{x-1}$

Solution

a. The degree of the numerator is less than the degree of the denominator, therefore $y = 0$ is the horizontal asymptote.

b. The numerator and the denominator both have the same degree, therefore the line $y = 1$ is the horizontal asymptote of the graph of g.

c. The degree of the numerator is larger than the degree of the denominator, therefore the graph of h has no horizontal asymptote.

Oblique Asymptote

The rational function $f(x) = \dfrac{p(x)}{q(x)}$ in lowest terms will have an oblique asymptote, if the degree of the polynomial in numerator is one more than the degree of the polynomial in denominator. To find it divide the numerator by the denominator, and disregard any remainder. Set the quotient equal to y to get the equation of the asymptote.

Example 20 **Find the Oblique Asymptote of a Rational Function**

Find the oblique asymptote of each rational function.

a. $f(x) = \dfrac{x^2+1}{x-3}$ **b.** $g(x) = \dfrac{x^3+2x-3}{x-2}$

Solution

a. Since the numerator is of degree exactly one more than the denominator, there will be an oblique (slant) asymptote. To find the asymptote, divide x^2+1 by $x-3$.

$$\begin{array}{r} x+3 \\ x-3\,\overline{)\,x^2+0x+1} \\ \underline{x^2-3x} \\ 3x+1 \\ \underline{3x-9} \\ 10 \end{array}$$

Therefore,

$$f(x) = \frac{x^2+1}{x-3} = x-3+\frac{10}{x+3}$$

and the line $y = x-3$ is an **oblique asymptote.**

b. Since the degree of the numerator is not exactly one more than the degree of the denominator, this rational function does not have an oblique asymptote.

Graphs of Rational Functions

Procedures for Graphing a Rational Function

To graph a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

written in lowest terms, apply the following steps:

1. Find the $x - intercepts$, if any, by solving $p(x) = 0$.

2. Find the $y - intercept$, by evaluating $f(0)$.

3. Find the vertical asymptotes.

4. Locate any horizontal or oblique asymptotes.

5. Determine points if any at which the graph of f intersects its nonvertical asymptotes by solving $f(x) = k$, where k is the $y - value$ of the nonvertical asymptote.

6. Test for symmetry.

- If $f(-x) = f(x)$, there is symmetry with respect to the $y - axis$.
- If $f(-x) = -f(x)$, there is symmetry with respect to the *origin*.

7. Determine where the graph is below or above the $x - axis$, using test numbers in each interval of the domain determined by the $x - intercepts$ and vertical asymptotes.

8. Use all the information obtained above and complete the sketch.

Remark: If the numerator and denominator of a rational function have common factors, begin by reducing the rational function to lowest terms. The graph of f is the same as the graph of the new function that is formed after the reducing process, except it has "holes". This is an indication that f is not a continuous function.

Example 21 **Graph a Rational Function**

Sketch the graph of $f(x) = \dfrac{x^2+4}{x^2-4}$

Solution

Step 1: To find the $x - intercept$ set the numerator of $f(x)$ equal to zero.

$$x^2 + 4 = 0$$

Since solution of the above quadratic equation is not real, there is no $x - intercept$.

Step 2: To find the $y - intercept$ find $f(0)$.

$$f(0) = \frac{0+4}{0-4} = -1$$

The $y - intercept$ is $(0, -1)$.

Step 3: To find the vertical asymptotes, set the denominator equal to zero.

$$x^2 - 4 = 0$$

$$x = \pm 2 \leftarrow \textbf{vertical asymptotes}$$

Step 4: Since the degree of the numerator is the same as the degree of the denominator, $y = 1$ is the **horizontal asymptote.**

Step 5: Let

$$f(x) = 1 \leftarrow \textbf{value of the horizontal asymptote}$$

$$\frac{x^2+4}{x^2-4} = 1$$

$$(\mathrm{x}^2 - 4)\frac{x^2+4}{x^2-4} = (\mathrm{x}^2 - 4)$$

$$x^2 + 4 = x^2 - 4$$

$$4 \neq -4$$

Thus, the graph of f does not intersect the horizontal asymptote.

Step 6: As shown below $f(x) = f(-x)$. Thus, the function f is an even function and its graph is symmetric with respect to the $y - axis$.

$$f(-x) = \frac{(-x)^2 + 4}{(-x)^2 - 4} = \frac{x^2 + 4}{x^2 - 4} = f(x)$$

Step 7:

	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
x	-3	1	3
f	$\frac{13}{5}$	$-\frac{5}{3}$	$\frac{13}{5}$

Step 8:

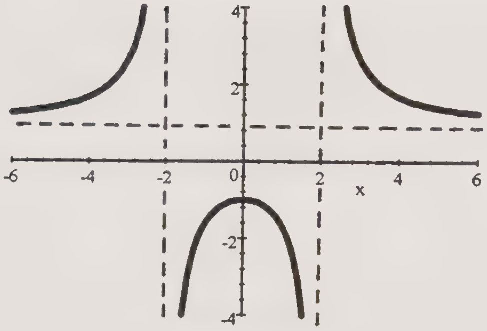


Figure 6.3

SOLVED PROBLEMS

Graphs of Polynomial Functions

For each polynomial function list the real zeros and state their multiplicity.

1. $f(x) = 2(x+2)^2(x-5)(x-7)^3$

Solution

To find zeros of the polynomial, set each parentheses equal to zero. Thus,

-2 is a zero of multiplicity 2.
5 is a zero of multiplicity 1.
7 is a zero of multiplicity 3.

2. $f(x) = 3(x^2+4)(x-2)^3$

Solution

To find zeros set each parentheses equal to zero and solve for x. Therefore,

$$x^2+4=0 \quad \text{or} \quad (x-2)^3=0$$
$$x=\pm 2i \qquad\qquad x=2$$

This function has one real zero, and that is 2, with multiplicity of 3.

3. $f(x) = (2x-3)^3(x+2)$

Solution

Set each parentheses equal to zero, and solve for x.

$$2x-3=0 \quad \text{or} \quad x+2=0$$
$$x=\tfrac{3}{2} \qquad\qquad x=-2$$

$\frac{3}{2}$ is a zero of multiplicity 3.

-2 is a zero of multiplicity 1.

Sketch the graph of each polynomial function.

4. $f(x) = (3-x)(x^2-5x+6)$

Solution

$f(x) = (3-x)(x^2-5x+6)$
$f(x) = (3-x)(x-3)(x-2)$
$f(x) = -(x-3)(x-3)(x-2)$
$f(x) = -(x-3)^2(x-2)$

Step 1: The $y-intercept$ is $f(0)=18$. The $x-intercepts$ satisfy the equation

$$f(x) = -(x-3)^2(x-2) = 0$$

Thus,

$$(x-3)^2=0 \quad \text{or} \quad x-2=0$$
$$x=3 \qquad\qquad x=2$$

The $x-intercepts$ are 3 and 2.

Step 2: The $x-intercept$, 3, is a zero of even multiplicity, so the graph of f touches the $x-axis$.

The $x-intercept$, 2, is a zero of odd multiplicity, so the graph of f crosses the $x-axis$.

Step 3: The maximum number of turning points is 2.

Steps 4&5:

	$-\infty < x < 2$	$2 < x < 3$	$3 < x < \infty$
x	1	$\frac{5}{2}$	4
f	4	$-\frac{1}{8}$	-2

Step 6:

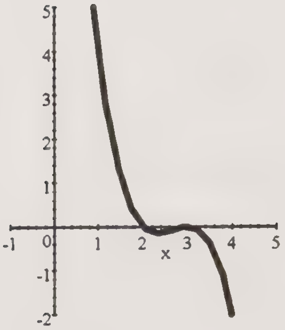


Figure 6.4

5. $f(x) = x^2 - x - 6$

Solution

$f(x) = x^2 - x - 6$

$f(x) = (x+2)(x-3)$

Step 1: The $y-intercept$ is $f(0) = -6$. The $x-intercepts$ satisfy the equation

$$f(x) = (x+2)(x-3) = 0$$

Thus,

$$x + 2 = 0 \quad or \quad x - 3 = 0$$

$$x = -2 \qquad\qquad x = 3$$

The $x - intercepts$ are -2 and 3.

Step 2: The $x - intercept$, -2, is a zero of odd multiplicity, so the graph of f crosses the $x - axis$.

The $x - intercept$, 3, is a zero of odd multiplicity, so the graph of f crosses the $x - axis$.

Step 3: The maximum number of turning points is 1.

Steps 4&5:

	$-\infty < x < -2$	$-2 < x < 3$	$3 < x < \infty$
x	-3	1	4
f	6	-6	6

Step 6:

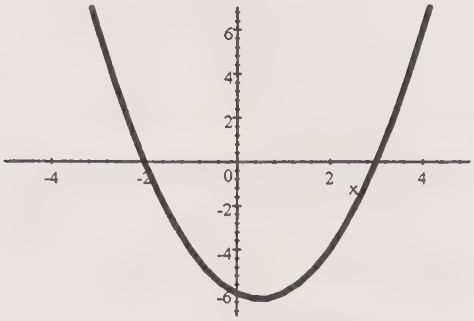


Figure 6.5

6. $f(x) = (x-1)(x+2)(x-3)$

Solution

Step 1: The $y-intercept$ is $f(0) = 6$. The $x-intercepts$ satisfy the equation

$$f(x) = (x-1)(x+2)(x-3) = 0$$

Thus,

$$x - 1 = 0 \quad or \quad x + 2 = 0 \quad or \quad x - 3 = 0$$
$$x = 1 \qquad\qquad x = -2 \qquad\qquad x = 3$$

Step 2: The $x - intercept$, 1, is a zero of odd multiplicity, so the graph of f crosses $x - axis$.

The $x - intercept$, –2, is a zero of odd multiplicity, so the graph of f crosses $x - axis$.

The $x - intercept$, 3, is a zero of odd multiplicity, so the graph of f crosses $x - axis$.

Step 3: The maximum number of turning points is 2.

Steps 4&5:

	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < 3$	$3 < x < \infty$
x	-3	-1	2	4
f	-24	8	-4	18

Step 6:

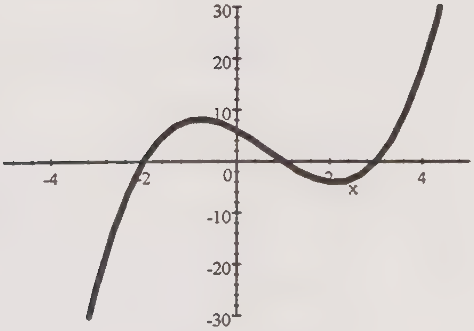


Figure 6.6

The Remainder and Factor Theorem

Use the Remainder Theorem to find the remainder in each of the following divisions.

7. $(x^3 - 2x^2 + 5x - 4) \div (x + 2)$

Solution

When the polynomial $f(x) = x^3 - 2x^2 + 5x - 4$ is divided by $x + 2$ or $x - (-2)$, the remainder is

$$\begin{aligned} r = f(-2) &= (-2)^3 - 2(-2)^2 + 5(-2) - 4 \\ &= -8 - 8 - 10 - 4 \\ &= -30 \end{aligned}$$

The remainder is -30.

8. $(x^4 - 5x^3 + 4x^2 + 8) \div (x - 2)$

Solution

When the polynomial $f(x) = x^4 - 5x^3 + 4x^2 + 8$ is divided by $x - 2$, the remainder is

$$\begin{aligned} r = f(2) &= (2)^4 - 5(2)^3 + 4(2)^2 + 8 \\ &= 16 - 40 + 16 + 8 \\ &= 0 \end{aligned}$$

The remainder is 0.

Use synthetic division and the Remainder Theorem to find $f(c)$.

9. $f(x) = 2x^4 - 3x^2 + x - 5,\ c = 2$

Solution

To find $f(2)$, we divide $f(x)$ by $x - 2$.

$$\begin{array}{r|rrrrr} 2) & 2 & 0 & -3 & 1 & -5 \\ & & 4 & 8 & 10 & 22 \\ \hline & 2 & 4 & 5 & 11 & 17 \end{array} \quad \leftarrow f(2)$$

The remainder, 17, is the value of $f(2)$. Thus, $f(2) = 17$.

10. $f(x) = -x^3 - 4x^2 + 2,\ c = -2$

Solution

$$\begin{array}{r|rrrr} -2) & -1 & -4 & 0 & 2 \\ & & 2 & 4 & -8 \\ \hline & -1 & -2 & 4 & -6 \end{array} \quad \leftarrow f(-2)$$

The remainder, -6, is the value of $f(-2)$. Thus, $f(-2) = -6$.

Use synthetic division to show that c is a zero of f.

11. $f(x) = 4x^3 - 10x^2 + x + 6,\ c = 2$

Solution

Perform synthetic division and examine the remainder.

$$\begin{array}{r|rrrr} 2) & 4 & -10 & 1 & 6 \\ & & 8 & -4 & -6 \\ \hline & 4 & -2 & -3 & 0 \end{array} \quad \leftarrow \textit{remainder}$$

Since remainder is zero, $c = 2$ is a zero of f.

12. $f(x) = 2x^5 - 4x^3 - 4x^2 + 2,\ c = -1$

Solution

$$\begin{array}{r|rrrrrr} -1 & 2 & 0 & -4 & -4 & 0 & 2 \\ & & -2 & 2 & 2 & 2 & -2 \\ \hline & 2 & -2 & -2 & -2 & 2 & 0 \end{array} \quad \leftarrow \textit{remainder}$$

The remainder is zero, thus $c = -1$ is a zero of f.

Use synthetic division to determine whether $g(x)$ *is a factor of* $f(x)$.

13. $f(x) = x^3 + 2x^2 - 6x + 5, \quad g(x) = x - 2$

Solution

Perform synthetic division and examine the remainder.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -6 & 5 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & 9 \end{array} \quad \leftarrow \textit{remainder}$$

The remainder, 9, implies that $x - 2$ is not a factor of f.

14. $f(x) = x^4 + 3x^3 - 5x^2 - x + 12, \quad g(x) = x + 4$

Solution

$$\begin{array}{r|rrrrr} -4 & 1 & 3 & -5 & -1 & 12 \\ & & -4 & 4 & 4 & -12 \\ \hline & 1 & -1 & -1 & 3 & 0 \end{array} \quad \leftarrow \textit{remainder}$$

The remainder, 0, implies that $x + 4$ is a factor of f.

Zeros of Polynomial Functions

Use the Rational Zero Theorem to list all possible rational zeros of each polynomial.

15. $f(x) = 3x^3 + 2x^2 - 3x - 6$

Solution

List all integers p that are factors of -6 and all integers q that are factors of 3.

$p: \quad \pm 2, \pm 3, \pm 6, \pm 1$

$q: \quad \pm 3, \pm 1$

Form all possible rational numbers using factors of p and q. Thus,

$$\frac{p}{q}: \quad \frac{\pm 2}{\pm 3}, \frac{\pm 2}{\pm 1}, \frac{\pm 3}{\pm 3}, \frac{\pm 3}{\pm 1}, \frac{\pm 6}{\pm 3}, \frac{\pm 6}{\pm 1}, \frac{\pm 1}{\pm 3}, \frac{\pm 1}{\pm 1}$$

The possible rational zeros are

$$\frac{p}{q}: \quad \pm\frac{2}{3}, \pm 2, \pm 1, \pm 3, \pm 6, \pm\frac{1}{3}$$

16. $f(x) = x^4 + 5x^3 - 2x^2 + 5$

Solution

$p: \quad \pm 1, \pm 5$

$q: \quad \pm 1$

The possible rational zeros are

$$\frac{p}{q}: \quad \pm 1, \pm 5$$

Use Descartes' Rule of Signs to state the number of possible positive and negative real zeros of each polynomial.

17. $f(x) = 6x^4 - 2x^3 + 3x^2 - 9$

Solution

$$f(x) = 6x^4 - 2x^3 + 3x^2 - 9$$

There are three variations in sign of $f(x)$. By Descartes' Rule of Signs there are either three or one positive real zeros.

$$f(-x) = 6(-x)^4 - 2(-x)^3 + 3(-x)^2 - 9$$
$$f(-x) = 6x^4 + 2x^3 + 3x^2 - 9$$

There is one variation in sign of $f(-x)$. By Descartes' Rule of Signs there is one negative real zero.

18. $f(x) = 3x^5 + 2x^4 + 3x^2 + 5$

Solution

$$f(x) = 3x^5 + 2x^4 + 3x^2 + 5$$

There is no variation in sign of $f(x)$. By Descartes' Rule of Signs there are no positive real zeros.

$$f(-x) = 3(-x)^5 + 2(-x)^4 + 3(-x)^2 + 5$$
$$f(-x) = -x^5 + 2x^4 + 3x^2 + 5$$

There is one variation in sign of $f(-x)$. Thus, there is one negative real zero.

Find all zeros of each polynomial and write the polynomial in a reduced form.

19. $f(x) = x^3 + 2x^2 - 13x + 10$

Solution

Step 1: The maximum number of zeros is three.

Step 2: All rational zeros of f are of the form $\frac{p}{q}$, where p is all factors of 10 and q is all factors of 1.

$p: \quad \pm1, \pm10, \pm2, \pm5$
$q: \quad \pm1$

$\frac{p}{q}: \quad \pm1, \pm10, \pm2, \pm5$

The possible rational zeros are ±1, ±10, ±2, ±5.

Step 3:

$$f(x) = x^3 + 2x^2 - 13x + 10$$

There are two variations in sign of $f(x)$. Thus, there are either 2 or no positive real zeros.

$$f(-x) = (-x)^3 + 2(-x)^2 - 13(-x) + 10$$
$$f(-x) = -x^3 + 2x^2 + 13x + 10$$

There is one variation in sign of $f(x)$. Thus, there is one negative real zero.

Step 4: Use synthetic division to test $x = 1$

$$\begin{array}{r|rrrr} 1) & 1 & 2 & -13 & 10 \\ & & 1 & 3 & -10 \\ \hline & 1 & 3 & -10 & 0 \end{array} \quad \leftarrow \textit{remainder}$$

Since the remainder is zero, 1 is a rational zero of f and hence $x - 1$ is a factor of $f(x)$.

Step 5: Function f can be written as

$$f(x) = (x-1)\left(x^2+3x-10\right)$$

By factoring $x^2+3x-10$ as

$$x^2+3x-10 = (x+5)(x-2)$$

we obtain

$$f(x) = (x-1)(x+5)(x-2)$$

Rational zeros of f are 1, -5, and 2.

20. $f(x) = 2x^3 - 7x^2 - 5x + 4$

Solution

Step 1: The maximum number of zeros is three.

Step 2: Rational zeros of f are of the form $\frac{p}{q}$, where p is all factors of 4 and q is all factors of 2.

$$\begin{array}{ll} p: & \pm1, \pm4, \pm2 \\ q: & \pm1, \pm2 \end{array}$$

$$\frac{p}{q}: \quad \pm1, \pm4, \pm2, \pm\frac{1}{2}$$

The possible rational zeros are $\pm1, \pm4, \pm2, \pm\dfrac{1}{2}$.

Step 3:

$$f(x) = 2x^3 - 7x^2 - 5x + 4$$

There are two variations in sign of $f(x)$. Thus, there are either 2 or no positive real zeros.

$$f(x) = 2(-x)^3 - 7(-x)^2 - 5(-x) + 4$$
$$f(-x) = -2x^3 - 7x^2 + 5x + 4$$

There is one variation in sign of $f(-x)$. Thus, there is one negative real zero.

Step 4: Use synthetic division to test potential zeros.

Test $x = 1$

$$\begin{array}{r|rrrr} 1) & 2 & -7 & -5 & 4 \\ & & 2 & -5 & -10 \\ \hline & 2 & -5 & -10 & \mathbf{-6} \end{array} \quad \leftarrow \textit{remainder}$$

The remainder is -6, thus 1 is not a zero of $f(x)$.

Test $x = -1$

$$\begin{array}{r|rrrr} -1) & 2 & -7 & -5 & 4 \\ & & -2 & 9 & -4 \\ \hline & 2 & -9 & 4 & 0 \end{array} \quad \leftarrow \textit{remainder}$$

Since the remainder is zero, -1 is a zero of $f(x)$. Therefore, $x+1$ is a factor of $f(x)$.

Step 5:

$$f(x) = (x+1)\left(2x^2 - 9x + 4\right)$$

By factoring $2x^2 - 9x + 4$ as

$$2x^2 - 9x + 4 = (2x-1)(x-4)$$

we obtain

$$f(x) = (x+1)(2x-1)(x-4)$$
$$f(x) = 2(x+1)\left(x - \frac{1}{2}\right)(x-4)$$

The rational zeros of function f are $-1, \dfrac{1}{2}, 4$.

21. $f(x) = 3x^4 + 2x^3 - 21x^2 - 32x - 12$

Solution

Step 1: The maximum number of zeros is four.

Step 2:

$$\begin{array}{ll} p: & \pm 1, \pm 2, \pm 12, \pm 6, \pm 2, \pm 3, \pm 4 \\ q: & \pm 1, \pm 3 \\ \dfrac{p}{q}: & \pm 1, \pm \dfrac{1}{3}, \pm 2, \pm \dfrac{2}{3}, \pm 12, \pm 4, \pm 6, \pm 3, \pm \dfrac{4}{3} \end{array}$$

The possible rational zeros are $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm 3, \pm 4, \pm 6, \pm 12.$

Step 3:

$$f(x) = 3x^4 + 2x^3 - 21x^2 - 32x - 12$$

There is one positive real zero.

$$f(-x) = 3(-x)^4 + 2(-x)^3 - 21(-x)^2 - 32(-x) - 12$$
$$f(-x) = 3x^4 - 2x^3 - 21x^2 + 32x - 12$$

There are either 3 or 1 negative real zeros.

Step 4: Use synthetic division to test zeros.

Test $x = 1$

$$\begin{array}{r|rrrrr} 1) & 3 & 2 & -21 & -32 & -12 \\ & & 3 & 5 & -16 & -48 \\ \hline & 3 & 5 & -16 & -48 & -60 \end{array} \leftarrow \textit{remainder}$$

Test $x = -1$

$$\begin{array}{r|rrrrr} -1) & 3 & 2 & -21 & -32 & -12 \\ & & -3 & 1 & 20 & 12 \\ \hline & 3 & -1 & -20 & -12 & 0 \end{array} \leftarrow \textit{remainder}$$

-1 is a zero of $f(x)$. Thus, $x + 1$ is a factor of f.

Step 5:

$$f(x) = (x+1)\left(3x^3 - x^2 - 20x - 12\right)$$

Now, we will find the zeros of $3x^3 - x^2 - 20x - 12$.

Test $x = -2$

$$\begin{array}{r|rrrr} -2) & 3 & -1 & -20 & -12 \\ & & -6 & 14 & 12 \\ \hline & 3 & -7 & -6 & 0 \end{array} \leftarrow \textit{remainder}$$

-2 is a zero of $3x^3 - x^2 - 20x - 12$. Thus, $x + 2$ is a factor of f.

$$f(x) = (x+1)(x+2)\left(3x^2 - 7x - 6\right)$$

By factoring $3x^2 - 7x - 6$ as

$$3x^2 - 7x - 6 = (3x+2)(x-3)$$

we obtain

$$f(x) = (x+1)(x+2)(3x+2)(x-3)$$

$$f(x) = 3(x+1)(x+2)\left(x + \frac{2}{3}\right)(x-3)$$

The rational zeros of $f(x)$ are $-1, -2, -\frac{2}{3}, 3$.

Upper and Lower Bound Rule

Find the smallest positive integer and the largest negative integer that by the Upper and Lower Bound Rule are upper and lower bounds for the real zeros of the following polynomials.

22. $f(x) = x^3 + 2x^2 - 13x + 10$

Solution

a. Test 1, 2, 3, 4, . . . as possible upper bounds, using synthetic division.

$$\begin{array}{r|rrrr} 1) & 1 & 2 & -13 & 10 \\ & & 1 & 3 & -10 \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2) & 1 & 2 & -13 & 10 \\ & & 2 & 8 & -10 \\ \hline & 1 & 4 & -5 & 0 \end{array}$$

$$\begin{array}{r|rrrrl} 3) & 1 & 2 & -13 & 10 & \\ & & 3 & 15 & 6 & \\ \hline & 1 & 5 & 2 & 16 & \leftarrow \textbf{All positive signs} \end{array}$$

Since $r = 3 > 0$ and all numbers in the third row of the synthetic division are positive, it can be concluded that 3 is an upper bound for the zeros of f.

b. Test $-1, -2, -3, \ldots$ as possible lower bounds, using synthetic division.

$$\begin{array}{r|rrrr} -1) & 1 & 2 & -13 & 10 \\ & & -1 & -1 & 14 \\ \hline & 1 & 1 & -14 & 24 \end{array}$$

$$\begin{array}{r|rrrr} -2) & 1 & 2 & -13 & 10 \\ & & -2 & 0 & 26 \\ \hline & 1 & 0 & -13 & 36 \end{array}$$

$$\begin{array}{r|rrrr} -3) & 1 & 2 & -13 & 10 \\ & & -3 & 3 & 30 \\ \hline & 1 & -1 & -10 & 40 \end{array}$$

$$\begin{array}{r|rrrr} -4) & 1 & 2 & -13 & 10 \\ & & -4 & 8 & 20 \\ \hline & 1 & -2 & -5 & 30 \end{array}$$

$$\begin{array}{r|rrrrl} -5) & 1 & 2 & -13 & 10 & \\ & & -5 & 15 & -10 & \\ \hline & 1 & -3 & 2 & 0 & \leftarrow \textbf{Alternating signs} \end{array}$$

Since $r = -5 < 0$ and all numbers in the third row of the synthetic division alternate in sign, it can be said that -5 is a lower bound for the zeros of f.

Thus,

$$-5 \le zeros\ of\ f \le 3$$

It can be concluded that all zeros of f lie in the interval $[-5, 3]$.

23. $f(x) = 2x^3 - 7x^2 - 5x + 4$

Solution

a. Test $1, 2, 3, \ldots$ as possible upper bounds.

$$\begin{array}{r|rrrr} 1 & 2 & -7 & -5 & 4 \\ & & 2 & -5 & -10 \\ \hline & 2 & -5 & -10 & -6 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 2 & -7 & -5 & 4 \\ & & 4 & -6 & -22 \\ \hline & 2 & -3 & -11 & -18 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & -5 & 4 \\ & & 6 & -3 & -24 \\ \hline & 2 & -1 & -8 & -20 \end{array}$$

$$\begin{array}{r|rrrr} 4 & 2 & -7 & -5 & 4 \\ & & 8 & 4 & -4 \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 5 & 2 & -7 & -5 & 4 \\ & & 10 & 15 & 50 \\ \hline & 2 & 3 & 10 & 54 \end{array} \longleftarrow \textbf{All positive numbers}$$

The synthetic divisor $r = 5 > 0$ and all numbers in the third row of the synthetic division are positive, therefore 5 is an upper bound for the zeros of f.

b. Test $-1, -2, -3, \ldots$ as possible lower bounds.

$$\begin{array}{r|rrrr} -1 & 2 & -7 & -5 & 4 \\ & & -2 & 9 & -4 \\ \hline & 2 & -9 & 4 & 0 \end{array} \longleftarrow \textbf{Alternating signs}$$

The synthetic divisor $r = -1 < 0$ and all numbers in the third row of the synthetic division are alternating in sign, therefore -1 is a lower bound.

$$-1 \leq \text{zeros of } f \leq 5$$

Thus all zeros of $f(x)$ lie in the interval $[-1, 5]$.

24. $f(x) = 3x^4 - 8x^3 + 3x^2 + 2x$

Solution

a. Test $1, 2, 3, \ldots$ as possible upper bounds.

$$\begin{array}{r|rrrrr} 1 & 3 & -8 & 3 & 2 & 0 \\ & & 3 & -5 & -2 & 0 \\ \hline & 3 & -5 & -2 & 0 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 3 & -8 & 3 & 2 & 0 \\ & & 6 & -4 & -2 & 0 \\ \hline & 3 & -2 & -1 & 0 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 3 & 3 & -8 & 3 & 2 & 0 \\ & & 9 & 3 & 18 & 60 \\ \hline & 3 & 1 & 6 & 20 & 60 \end{array} \longleftarrow \textbf{All positive numbers}$$

Since the synthetic divisor $r = 3 > 0$ and all the numbers in the third row of the synthetic division are positive, it is concluded that 3 is an upper bound.

b. Test $-1, -2, -3, \ldots$ as possible lower bounds..

$$\begin{array}{r|rrrrr} -1 & 3 & -8 & 3 & 2 & 0 \\ & & -3 & 11 & -14 & 12 \\ \hline & 3 & -11 & 14 & -12 & 12 \end{array} \longleftarrow \textbf{Alternating signs}$$

Since the synthetic divisor $r = -1 < 0$ and the numbers in the third row of the synthetic division are alternating in sign, -1 is a lower bound.

Thus, all zeros of $f(x)$ lie in the interval $[-1, 3]$.

Complex Polynomial Functions

Write each polynomial as a product of linear factors.

25. $f(x) = x^2 + 4$

Solution

Factor $f(x)$ as follows:

$$f(x) = x^2 + 4 = (x + 2i)(x - 2i)$$

26. $f(x) = x^3 + 25x$

Solution:

$$\begin{aligned} f(x) &= x^3 + 25x \\ &= x\left(x^2 + 25\right) \\ &= x(x + 5i)(x - 5i) \end{aligned}$$

27. $f(x) = x^3 - 2x^2 - 3x$

Solution

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 3x \\ &= x\left(x^2 - 2x - 3\right) \\ &= x(x - 3)(x + 1) \end{aligned}$$

28. $f(x) = x^4 - 2x^3 + 6x^2 - 18x - 27$

Solution

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 6x^2 - 18x - 27 \\ f(x) &= \left(x^4 + 6x^2 - 27\right) - 2x^3 - 18x \\ &= \left(x^2 + 9\right)\left(x^2 - 3\right) - 2x\left(x^2 + 9\right) \\ &= \left(x^2 + 9\right)\left(x^2 - 3 - 2x\right) \\ f(x) &= (x + 3i)(x - 3i)(x - 3)(x + 1) \end{aligned}$$

Find a polynomial given its zeros.

29. $2, -5, 3$

Solution

Since $2, -5$, and 3 are zeros of a polynomial, $(x - 2)$, $[x - (-5)]$, and $(x - 3)$ are factors of the polynomial.

Thus,

$$\begin{aligned} f(x) &= (x - 2)(x + 5)(x - 3) \\ &= \left(x^2 + 3x - 10\right)(x - 3) \\ &= x^3 - 19x + 30 \end{aligned}$$

The polynomial is

$$f(x) = x^3 - 19x + 30$$

30. $2 + i, 2 - i, 4$

Solution

Since $2+i, 2-i$, and 4 are zeros of a polynomial, then $[x - (2 + i)]$, $[x - (2 - i)]$, and $(x - 4)$ are factors of the polynomial.

Thus,

$$\begin{aligned} f(x) &= [x - (2 + i)][x - (2 - i)][x - 4] \\ &= (x - 2 - i)(x - 2 + i)(x - 4) \end{aligned}$$

$$\begin{aligned}
&= \left[(x-2)^2 - i^2\right](x-4)\\
&= \left[(x-2)^2 + 1\right](x-4)\\
&= \left(x^2 - 4x + 4 + 1\right)(x-4)\\
&= x^3 - 8x^2 + 21x - 20
\end{aligned}$$

The polynomial is

$$f(x) = x^3 - 8x^2 + 21x - 20$$

31. Write $f(x) = x^4 - 8x^3 + 23x^2 - 30x + 18$ as a product of linear factors, given that 3 is a zero of multiplicity 2.

Solution

$$f(x) = (x-3)^2\, g(x)$$

Now $g(x)$ can be found using synthetic division twice. Thus,

$$\begin{array}{r|rrrrr}
3) & 1 & -8 & 23 & -30 & 18\\
 & & 3 & -15 & 24 & -18\\
\hline
3) & 1 & -5 & 8 & -6 & 0\\
 & & 3 & -6 & 6 & \\
\hline
 & 1 & -2 & 2 & 0 &
\end{array}$$

$g(x) = x^2 - 2x + 2$

or

$f(x) = (x-3)^2\left(x^2 - 2x + 2\right)$

The zeros of the quadratic function $g(x) = x^2 - 2x + 2$ are found using the quadratic formula.

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

By the factor theorem, polynomial $f(x)$ can be written as:

$$f(x) = (x-3)^2\,[x - (1+i)]\,[x - (1-i)]$$

The Conjugate Pair Theorem

32. Find all zeros of $f(x) = x^4 - 6x^3 + 14x^2 - 6x + 13$ given that $3 + 2i$ is a zero of f.

Solution

By the Conjugate Pair Theorem, since coefficients of f are all real and $3 + 2i$ is a zero of f, then $3 - 2i$ is also a zero of f. Using synthetic division with $3 + 2i$ and then the result with $3 - 2i$ we have

$$\begin{array}{r|rrrrr}
3+2i) & 1 & -6 & 14 & -6 & 13\\
 & & 3+2i & -13 & 3+2i & -13\\
\hline
3-2i) & 1 & -3+2i & 1 & -3+2i & 0\\
 & & 3-2i & 0 & 3-2i & \\
\hline
 & 1 & 0 & 1 & 0 &
\end{array}$$

Thus, the reduced polynomial is

$$f(x) = [x - (3+2i)]\,[x - (3-2i)]\,\left[x^2 + 1\right]$$

or

$$f(x) = [x - (3+2i)]\,[x - (3-2i)]\,[x - (i)]\,[x - (-i)]$$

zeros of f are $3 + 2i$, $3 - 2i$, i, $-i$.

33. Find all zeros of $f(x) = x^4 - 10x^3 + 35x^2 - 50x + 34$ given that $(1+i)$ is a zero of f.

Solution

$$\begin{array}{l|lllll}
1+i) & 1 & -10 & 35 & -50 & 34 \\
 & & 1+i & -10-8i & 33+17i & -34 \\ \hline
1-i) & 1 & -9+i & 25-8i & -17+17i & 0 \\
 & & 1-i & -8+8i & 17-17i & \\ \hline
 & 1 & -8 & 17 & 0 &
\end{array}$$

Thus, the reduced polynomial is

$$f(x) = [x-(1+i)]\,[x-(1-i)]\,g(x)$$

where

$$g(x) = x^2 - 8x + 17$$

Find zeros of $g(x)$, using the quadratic formula.

$$x^2 - 8x + 17 = 0$$

$$x = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2}$$

$$x = 4 \pm i$$

The reduced polynomial is

$$f(x) = [x-(1+i)]\,[x-(1-i)]\,[x-(4+i)]\,[x-(4-i)]$$

zeros of f are $1+i, 1-i, 4+i, 4-i$.

Asymptotes

Find the vertical asymptotes of each rational function.

34. $f(x) = \dfrac{x-4}{x^2-x-6}$

Solution

To find the vertical asymptotes, set the denominator equal to zero and solve for x.

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x+2 = 0 \quad \text{or} \quad x-3 = 0$$

$$x = -2 \qquad\qquad x = 3$$

Thus, $x = -2$ and $x = 3$ are **vertical asymptotes.**

35. $f(x) = \dfrac{x+2}{x^2+4}$

Solution

The denominator $x^2 + 4$ has no real zeros, thus the graph of f has no vertical asymptote.

Find the horizontal asymptote of each rational function.

36. $f(x) = \dfrac{x-2}{x^2+6}$

Solution

The degree of numerator is less than the degree of denominator, thus the line $y = 0$ is a **horizontal asymptote**.

37. $f(x) = \dfrac{2x^2+1}{5x^2-3}$

Solution

The degree of numerator is the same as the degree of denominator, thus the line $y = \frac{2}{5}$ is the **horizontal asymptote.**

38. $f(x) = \dfrac{x^3+1}{x+3}$

Solution

The degree of numerator is larger than the degree of the denominator, thus there is no horizontal asymptote.

Graphs of Rational Functions

DEGREE OF NUMERATOR < DEGREE OF DENOMINATOR

Sketch the graph of each rational function.

39. $f(x) = \dfrac{3}{2+x}$

Solution

Step 1: To find the $x-intercept$ set the numerator of $f(x)$ equal to zero. As $3 \neq 0$, thus there is no $x-intercept$.

Step 2: To find the $y-intercept$ find $f(0)$.

$$f(0) = \frac{3}{2+0} = \frac{3}{2}$$
$$y-intercept = \left(0, \frac{3}{2}\right)$$

Step 3: To find the vertical asymptotes set the denominator equal to zero and solve for x.

$$\begin{aligned} 2+x &= 0 \\ x &= -2 \qquad \textbf{vertical asymptote} \end{aligned}$$

Step 4: Since the numerator has lower degree than the denominator, $y = 0$ is a **horizontal asymptote.**

Step 5: To find the points at which the graph of f intersect its nonvertical asymptotes, set $f(x)$ equal to the $y-value$ of the nonvertical asymptote, and solve for x.

$$f(x) = 0 \leftarrow \textbf{value of the horizontal asymptote}$$

$$\frac{3}{2+x} = 0$$
$$(2+\mathbf{x})\frac{3}{2+x} = (2+\mathbf{x})0$$
$$3 \neq 0$$

Thus, the graph of f does not intersect the horizontal asymptote.

Step 6: There are no symmetries.

Step 7:

	$-\infty < x < -2$	$-2 < x < \infty$
x	-3	-1
f	-3	3

Step 8:

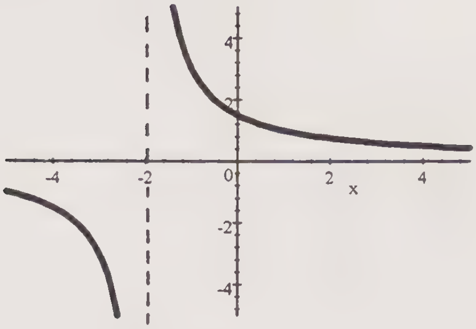


Figure 6.7

40. $f(x) = \dfrac{x+1}{(x-1)(x+3)}$

Solution

Step 1: To find the $x-intercept$ set the numerator of $f(x)$ equal to zero.

$$x+1=0$$
$$x=-1$$
$$x-intercept=(-1,0)$$

Step 2: To find the $y-intercept$, find $f(0)$.

$$f(0)=\frac{0+1}{(0-1)(0+3)}=-\frac{1}{3}$$
$$y-intercept=\left(0,-\frac{1}{3}\right)$$

Step 3: To find the vertical asymptotes, set the denominator equal to zero and solve for x.

$$(x-1)(x+3)=0$$
$$x-1=0 \quad \text{or} \quad x+3=0$$
$$x=1 \qquad x=-3$$

The **vertical asymptotes** are $x=1$ and $x=-3$.

Step 4: Since the degree of the numerator is lower than the degree of the denominator, $y=0$ is a **horizontal asymptote.**

Step 5: To determine whether the graph of f intersects its nonvertical asymptote, solve

$f(x)=0 \leftarrow$ **y-value of the horizontal asymptote**

$$0=\frac{x+1}{(x-1)(x+3)}$$
$$x+1=0$$
$$x=-1$$

The graph of f intersects the horizontal asymptote at $x=-1$.

Step 6: There are no symmetries.

Step 7:

	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
x	-4	-2	$\frac{1}{2}$	2
f	$-\frac{3}{5}$	$\frac{1}{3}$	$-\frac{6}{7}$	$\frac{3}{5}$

Step 8:

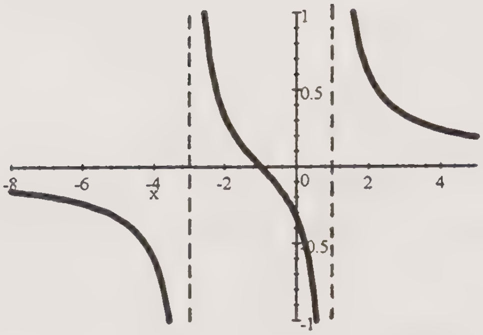


Figure 6.8

DEGREE OF NUMERATOR = DEGREE OF DENOMINATOR

Sketch the graph of each rational function.

41. $f(x) = \dfrac{2x^2}{x^2+2}$

Solution

Step 1: To find the $x-intercept$ set the numerator equal to zero.

$$2x^2 = 0$$
$$x = 0$$

The $x-intercept$ is $(0,0)$.

Step 2: To find the $y-intercept$ find $f(0)$.

$$f(0) = \frac{2(0)}{0+2} = 0$$

The $y-intercept$ is $(0,0)$.

Step 3: To find the vertical asymptotes, set the denominator equal to zero. Since the denominator has no real zeros, the graph of f has no vertical asymptotes.

Step 4: The numerator and denominator both have the same degree. Thus, $y = 2$ is the horizontal asymptote.

Step 5: To determine the points at which the graph of f intersects its nonvertical asymptotes, let

$$f(x) = 2 \leftarrow \textbf{value of the horizontal asymptote}$$

$$\frac{2x^2}{x^2+2} = 2$$

$$(x^2+2)\frac{2x^2}{x^2+2} = (x^2+2)\,2$$
$$2x^2 = 2x^2 + 4$$
$$0 \neq 4$$

Thus, the graph of f does not intersect the horizontal asymptote.

Step 6: As shown below since $f(x) = f(-x)$, the function f is an even function and therefore its graph is symmetric with respect to the $y-axis$.

$$f(-x) = \frac{2(-x)^2}{(-x)^2+2} = \frac{2x^2}{x^2+2} = f(x)$$

Step 7:

	$-\infty < x < 0$	$0 < x < \infty$
x	-1	1
f	$\frac{2}{3}$	$\frac{2}{3}$

Step 8:

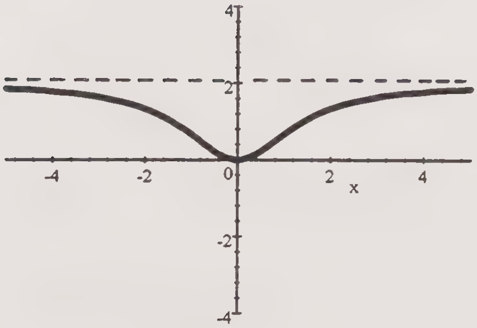


Figure 6.9

DEGREE OF NUMERATOR > DEGREE OF DENOMINATOR

Sketch the graph of the rational function.

42. $f(x) = \dfrac{x^2+2}{x-2}$

Solution

Step 1: $x^2 + 2 = 0$

There are no $x - intercepts$.

Step 2: $f(0) = \dfrac{0+2}{0-2} = -1$

The $y - intercept$ is $(0, -1)$.

Step 3: For vertical asymptotes set $x - 2$ equal to zero.

$x = 2 \leftarrow$ **vertical asymptote**

Step 4: There are no horizontal asymptotes. The degree of numerator is exactly one more than the degree of the denominator, so f has an oblique asymptote. Dividing $x^2 + 2$ by $x - 2$ shows that f can be expressed as

$$f(x) = \frac{x^2+2}{x-2} = x+2+\frac{6}{x-2}$$

Thus, $y = x + 2$ is an **oblique asymptote.**

Step 5: To determine whether the graph of f intersects its oblique asymptote, let

$$f(x) = \underbrace{x+2}_{\text{nonvertical asymptote}}$$

$$\frac{x^2+2}{x-2} = x+2$$

$$(\mathbf{x}-\mathbf{2})\frac{x^2+2}{x-2} = (\mathbf{x}-\mathbf{2})(x+2)$$

$$x^2+2 = x^2-4$$

$$2 \neq -4$$

Thus, graph of f does not intersect its oblique asymptote.

Step 6: There is no symmetry.

Step 7:

	$-\infty < x < 2$	$2 < x < \infty$
x	1	3
f	−3	11

Step 8:

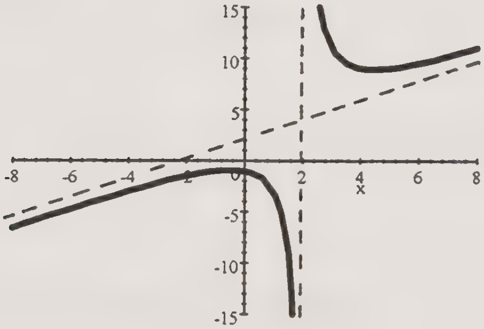


Figure 6.10

RATIONAL FUNCTIONS WITH COMMON FACTORS

Sketch the graph of each function.

43. $f(x) = \dfrac{x^2-2x-3}{x+1}$

Solution

Factor the numerator and denominator to obtain

$$f(x) = \frac{(x-3)(x+1)}{x+1}$$

reducing $f(x)$ to lowest terms yields

$$F(x) = x-3$$

Note: The functions f and F are not the same. However, they are equivalent for all values of $x \neq -1$. The domain of f is the set of all real numbers except -1, whereas the domain of F is the set of all real numbers. The graph of F is a straight line with slope 1 and $y - intercept$ of -3. The graph of f is the same as the graph of F with the exception of a hole at the point $(-1, -4)$ to indicate that f is undefined at this point.

The graph of f is

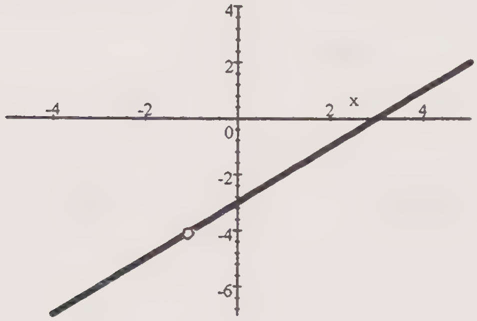


Figure 6.11

44. $f(x) = \dfrac{2x - 6}{x^2 - 2x - 3}$

Solution

Factor the numerator and denominator to obtain

$$f(x) = \frac{2(x-3)}{(x+1)(x-3)}$$
$$F(x) = \frac{2}{x+1}$$

Thus, the graph of f is the same as the graph of F, except that the graph of f will have a hole with an $x - value$ of 3.

$$F(x) = \frac{2}{3+1} = \frac{1}{2}$$

The hole is located at $\left(3, \frac{1}{2}\right)$.

To graph function F, proceed with the following steps.

Step 1: There are no $x - intercepts$.

Step 2: $F(0) = \dfrac{2}{0+1} = 2$

The $y - intercept$ is $(0, 2)$.

Step 3: The vertical asymptote is $x = -1$.

Step 4: The horizontal asymptote is $y = 0$.

Step 5: $F(x) = 0$

Thus,

$$0 = \frac{2}{x+1}$$
$$0 \neq 2$$

The graph of F does not intersect the horizontal asymptote.

Step 6: There are no symmetries.

Step 7:

	$-\infty < x < -1$	$-1 < x < \infty$
x	-2	1
F	-2	1

Step 8:

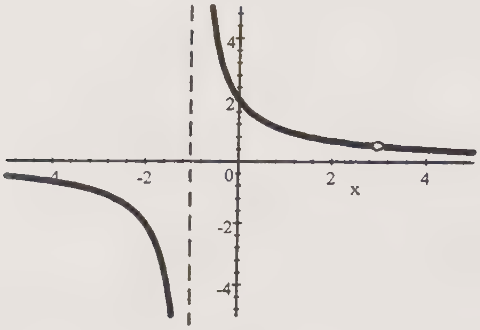


Figure 6.12

PRACTICE TEST

1. Use synthetic division to find quotient and remainder when $f(x)$ is divided by $g(x)$.

$$f(x) = 4x^3 - 2x^2 + 3x + 6 \qquad g(x) = x + 2$$

2. Use the Remainder Theorem to find remainder when $f(x)$ is divided by $g(x)$.

$$f(x) = x^4 + 2x + 4 \qquad g(x) = x - 2$$

3. Determine whether $(x - 2)$ and $(x + 1)$ are factors of

$$f(x) = 2x^3 - 5x^2 - 5x + 14$$

4. Sketch the graph of the polynomial defined by

$$P(x) = (x - 2)^2 (x + 2)^2$$

5. Use the Rational Zero Theorem to list the possible rational zeros for the polynomial

$$P(x) = 8x^3 - 2x^2 + 3x - 4$$

6. Find the zeros of $P(x) = (x^2 - 25)(x + 2)^2$, and state the multiplicity of each zero.

7. Use the Upper and Lower Bound Theorem, to find the smallest positive integer and the largest negative integer that are upper and lower bounds for the polynomial $P(x) = 3x^3 + 4x^2 - 8x + 5$.

8. Use Descartes' Rule of Signs to find the number of possible negative real zeros of the polynomial $f(x) = 7x^4 - 3x^3 - 4x^2 + 2x + 7$.

9. Given that i is a zero of

$$P(x) = x^3 - 3x^2 + x - 3$$

find the remaining zeros and write the polynomial as a product of linear factors.

10. Find all zeros of the polynomial $P(x) = x^3 - 2x^2 + 4x - 8$.

11. Find all vertical asymptotes of the graph of $f(x) = \dfrac{3x^2 - 5}{x^2 - 3x - 10}$.

12. Find all horizontal asymptotes of the graph of $f(x) = \dfrac{2x^2 + x + 1}{5x^2 - 1}$.

13. Graph: $f(x) = \dfrac{x - 2}{x^2 + 1}$

CHAPTER SEVEN

Exponential and Logarithmic Functions

7.1 Exponential Functions and their Graphs
7.2 Logarithmic Functions and their Graphs
7.3 Properties of Logarithms
7.4 Exponential and Logarithmic Equations
7.5 Applications of Exponential and Logarithmic Functions

7.1 EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

Definition The **exponential function** f with base b is defined by

$$f(x) = b^x$$

where b is a positive real number not equal to one and x is any real number

Exponential Functions

Features of the Exponential Functions

1. The domain of $f(x) = b^x$ is $(-\infty, +\infty)$, and the range is $(0, \infty)$.

2. The graphs pass through the point $(0, 1)$.

3. When $b > 1$, the graph goes up from left to right.
When $0 < b < 1$, the graph goes down from left to right.

4. The graphs get very close to the $x - axis$ but never cross it. The $x - axis$ is a **horizontal asymptote.**

5. The exponential function $f(x) = b^x$ is one-to-one. (This means it has an inverse).

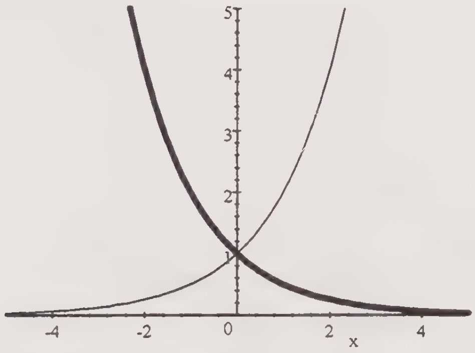


Figure 7.1

The Natural Base e

The function f defined by $f(x) = e^x$ is called the **natural exponential function.** The number e is defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59374246
100	2.704813829
$1,000$	2.716923932
$10,000$	2.718145926
$100,000$	2.718268237
$1,000,000$	2.718280469
$1,000,000,000$	2.718281828

For purpose of approximation use

$$e \approx 2.7$$

Example 1 **Graph Exponential Functions**

Sketch the graph of each function using graphing techniques. Also, find the domain and range of each function.

a. $f(x) = 2^x - 2$
b. $f(x) = 2^{-x} + 1$
c. $f(x) = e^{x-2}$

Solution

a. Use the following steps to get the graph of $f(x) = 2^x - 2$.
Step 1: $y = 2^x$ *Exponential function*
Step 2: $y = 2^x - 2$ *Vertical shift down 2 units*

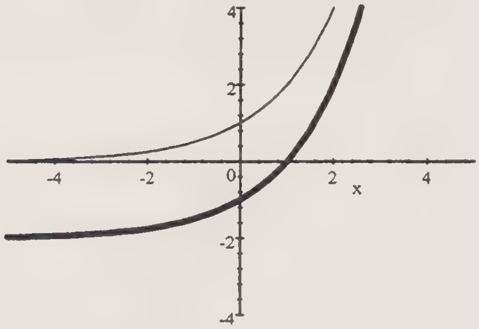


Figure 7.2

Domain $= (-\infty, \infty)$
Range $= (-2, \infty)$

b. Use the following steps to get the graph of $f(x) = 2^{-x} + 1$.
Step 1: $y = 2^x$ *Exponential function*
Step 2: $y = 2^{-x}$ *Reflect about y – axis*
Step 3: $y = 2^{-x} + 1$ *Vertical shift up 1 unit*

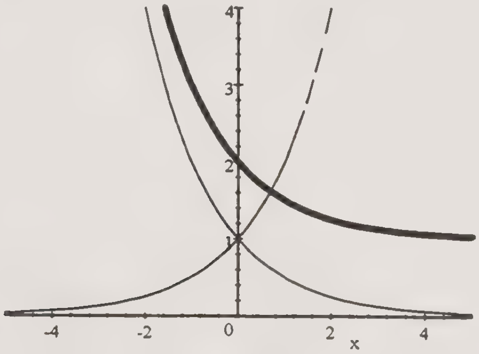


Figure 7.3

Domain $= (-\infty, \infty)$
Range $= (1, \infty)$

c. $f(x) = e^{x-2}$
Use the following steps to graph $f(x) = e^{x-2}$
Step 1: $y = e^x$ *Graph exponential function*
Step 2: $y = e^{x-2}$ *Horizontal shift 2 units to the right. If an exponential function is translated horizontally, the asymptote will not change. To find the y-intercept replace x with zero. Therefore, $y - intercept = e^{-2} \approx 0.14$.*

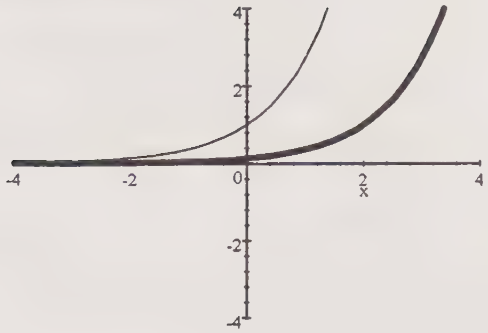


Figure 7.4

Domain = $(-\infty, \infty)$
Range = $(0, \infty)$

Note: If an exponential function is translated horizontally, the asymptote will not change.

7.2 LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

The one-to-one exponential function $y = f(x) = b^x$, $b > 0$, $b \neq 1$, has an inverse and it is defined by the equation

$$x = b^y \qquad b > 0, b \neq 1$$

The inverse is so important that it is given a name, the *logarithmic function.*

Definition If $b > 0$, $b \neq 1$, and $x > 0$, then

$$y = \log_b x \qquad \text{if and only if} \qquad x = b^y$$

In the equation $y = \log_b x$, x is the **argument,** and b is the **base.** The notation $\log_b x$ is read " log *of x to base b*"

Common Logarithm

The function defined by

$$f(x) = \log_{10} x = \log x, \qquad x > 0$$

is called the **common logarithm function.**

Natural Logarithm

The function defined by

$$f(x) = \log_e x = \ln x, \qquad x > 0$$

is called the **natural logarithm function.**

Evaluating Logarithms

EQUALITY OF EXPONENTS PROPERTY

To find the exact value of a logarithm, we write the logarithm in exponential notation and use the **equality of exponents property.**

If b is a positive real number $(b \neq 1)$ such that $b^x = b^y$, then $x = y$.

The above result is based on the fact that exponential functions are one-to-one.

Example 2 **Find the Exact Value of a Logarithmic Function**

Evaluate $\log_{\frac{1}{3}} \frac{1}{27}$.

Solution

Let $\log_{\frac{1}{3}} \frac{1}{27} = x$

$\left(\frac{1}{3}\right)^x = \frac{1}{27}$ • Change to exponential form.

$\left(\frac{1}{3}\right)^x = \left(\frac{1}{3}\right)^3$ • Factor.

$x = 3$ • Equality of exponents property.

CHANGE-OF-BASE FORMULA

Logarithms that can not be evaluated using the equality of exponents property, can be evaluated using the **change-of-base formula**.

If x, a, and b are positive real numbers with $a \neq 1$ and $b \neq 1$, then

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example 3 **Evaluate Logarithms Whose Base is Neither 10 nor e**

Evaluate $\log_4 24$.

Solution

$\log_4 24 = \frac{\log 24}{\log 4} \approx 2.2925$

Logarithmic Functions

Features of the Logarithmic Functions

1. The range of $y = \log_b x$ is $(-\infty, \infty)$, and the domain is $(0, \infty)$.

2. The graph of $y = \log_b x$ passes through the point $(1, 0)$.

3. When $b > 1$, the graph goes up from left to right.
When $0 < b < 1$, the graph goes down from left to right.

4. The graph gets very close to the $y - axis$, but never touch it. The $y - axis$ is a vertical asymptote.

5. The logarithmic function $y = \log_b x$ is one-to-one.

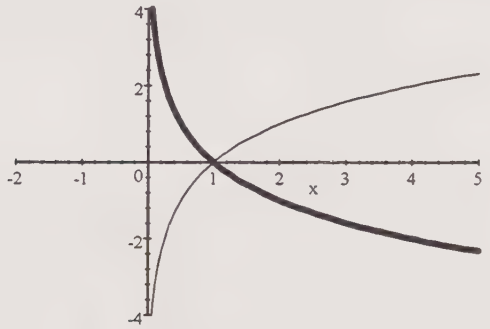


Figure 7.5

Example 4 **Graph Logarithmic Functions**

Sketch the graph of each function. Also, state the domain and range of each function.

a. $f(x) = \log_2 x - 2$

b. $f(x) = \log_3 (x + 2)$

Solution

a. Use the following steps to graph $f(x) = \log_2 x - 2$.

Step 1: $y = \log_2 x$ *Logarithmic function*

Step 2: $y = \log_2 x - 2$ *Vertical shift down by 2 units. If a logarithmic function is translated vertically, the asymptote will not change. To find the x-intercept, replace y with zero and solve for x.*

$\log_2 x = 2 \Longrightarrow x = 4$

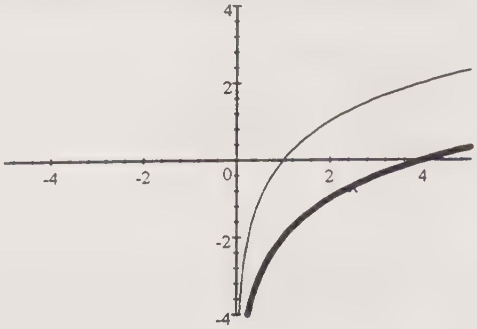


Figure 7.6

Note: If a logarithmic function is translated vertically, the asymptote will not change.

b. Use the following steps to graph $f(x) = \log_3(x+2)$.
Step 1: $y = \log_3 x$ *Logarithmic function*
Step 2: $y = \log_3(x+2)$ *Horizontal shift left 2 units*

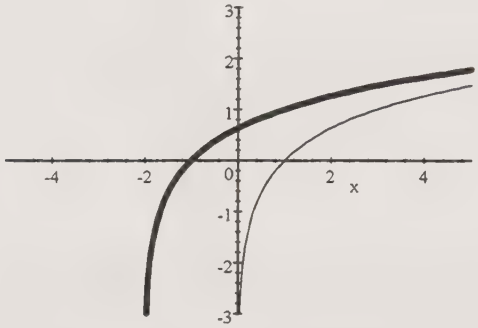


Figure 7.7

Domain of Logarithmic Functions

Example 5 **Find the Domain of a Logarithmic Function**

Find the domain of each logarithmic function.

a. $f(x) = \log_3(3-x)$

b. $g(x) = \log \dfrac{2x}{x^2-1}$

Solution

a. The domain of f consists of all x for which $(3-x) > 0$; that is all $x < 3$, or $(-\infty, 3)$.

b. The domain of g is restricted to

$$\frac{2x}{x^2-1} > 0$$

To solve this inequality, use the critical value method. The critical values include the zeros of the denominator, which are 1 and -1, and the zero of numerator, which is 0. These critical values divide the real number line into four intervals.

	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
x	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
$2x$	$-$	$-$	$+$	$+$
$x^2 - 1$	$+$	$-$	$-$	$+$
$\frac{2x}{x^2-1} > 0$	$-$	$+$	$-$	$+$

The above table indicates that the rational expression is positive on two intervals $(-1, 0)$ and $(1, \infty)$.
Therefore, the domain is $(-1, 0) \cup (1, \infty)$.

7.3 PROPERTIES OF LOGARITHMS

The logarithmic function with base b is the inverse of the exponential function with base b. Thus, it makes sense that the properties of exponents should have corresponding properties involving logarithms.

In the following properties, x, y, a and b are positive real numbers, with $a \neq 1$ and $b \neq 1$.

Properties of Logarithms used for Expanding Expressions

Product Rule	$\log_b (xy) = \log_b x + \log_b y$
Quotient Rule	$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Rule	$\log_b x^a = a \log_b x$

Properties of Logarithms used for Contracting Expressions

Product Rule	$\log_b x + \log_b y = \log_b (xy)$
Quotient Rule	$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$
Power Rule	$a \log_b x = \log_b x^a$

General Properties of Logarithms

1. $\log_b b = 1$
2. $\log_b 1 = 0$
3. $\log_b x$ is undefined if $x \leq 0$

4. $b^{\log_b x} = x$

5. $\log_b x = \dfrac{\log_a x}{\log_a b}$ **change of base rule**

6. $\log_b b^x = x$

CAUTION: Make note of the following correct and incorrect properties.

Correct	**Incorrect**
$\log_b (xy) = \log_b x + \log_b y$	$\log_b (x + y) = \log_b x + \log_b y$
$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\frac{\log_b x}{\log_b y} = \log_b x - \log_b y$
$\log_b x^a = a \log_b x$	$(\log_b x)^a = a \log_b x$

Example 6 **Rewrite Logarithmic Expressions**

Use the properties of logarithms to expand the following logarithms.

a. $\log x^2 \sqrt[3]{x^2 + 1}$

b. $\log \dfrac{x^2\sqrt{2x}}{(x+1)^3}$

Solution

a. $\log x^2\sqrt[3]{x^2+1} = \log x^2 + \log \sqrt[3]{x^2+1}$

$$= 2\log x + \tfrac{1}{3}\log\left(x^2+1\right)$$

b. $\log \dfrac{x^2\sqrt{2x}}{(x+1)^3} = \log x^2\sqrt{2x} - \log (x+1)^3$ • Quotient Rule.

$$= \log x^2 + \log\sqrt{2x} - \log(x+1)^3$$ • Product Rule

$$= 2\log x + \tfrac{1}{2}\log(2x) - 3\log(x+1)$$ • Power Rule.

$$= 2\log x + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\log x - 3\log(x+1)$$ • Product Rule.

$$= \tfrac{5}{2}\log x + \tfrac{1}{2}\log 2 - 3\log(x+1)$$

Example 7 **Rewrite Logarithmic Expressions**

Use the properties of logarithms to write the following expressions as a single logarithm.

a. $3\log_a(xy) - 2\log_a x - 2\log_a y$

b. $2\log(x+1) - \log\left(x^2-1\right) + \log(x-1)$

Solution

a. $3\log_a(xy) - 2\log_a x - 2\log_a y$

$$= \log_a(xy)^3 - \log_a x^2 - \log_a y^2$$

$$= \log_a x^3y^3 - \left(\log_a x^2 + \log_a y^2\right)$$

$$= \log_a x^3y^3 - \log_a x^2y^2$$

$$= \log_a \frac{x^3y^3}{x^2y^2}$$

$$= \log_a xy$$

b. $2\log(x+1) - \log\left(x^2-1\right) + \log(x-1)$

$= \log(x+1)^2 - \log\left(x^2-1\right) + \log(x-1)$

$$= \log\frac{(x+1)^2}{x^2-1} + \log(x-1)$$

$$= \log\frac{(x+1)^2}{x^2-1}\cdot(x-1)$$

$$= \log\frac{(x+1)^2}{(x+1)(x-1)}\cdot(x-1)$$

$$= \log(x+1)$$

7.4 EXPONENTIAL AND LOGARITHMIC EQUATIONS

Exponential Equations

Equations that involve terms of the form a^x, $a > 0$, $a \neq 1$, are referred to as **exponential equations.** Such equations can be divided into two categories:
Category 1 *Equations that can be written as a power of the same base.*
To solve equations of this nature apply the laws of exponents and property of exponential equations.

If $b^x = b^y$, then $x = y$, for $b > 0$ and $b \neq 1$.

Example 8 Solve an Exponential Equation

Solve: $4^x = 16$.
Solution
Write each side as a power of base 4
$$4^x = 4^2$$
Apply the property of exponential equations to obtain
$$x = 2$$
The solution set is $\{2\}$.

Example 9 Solve an Exponential Equation

Solve: $e^{x^2+1} = e^5 \cdot \frac{1}{e^{3x}}$.
Solution
Use the laws of exponents to get the same base on each side.
$$e^{x^2+1} = e^{5-3x}$$
Apply the property of exponential equations to get
$$x^2 + 1 = 5 - 3x$$
$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$$x = -4 \quad \text{or} \quad x = 1$$
The solution set is $\{-4, 1\}$.

Category 2 *Equations that can not be written as a power of the same base.*
When we are not able to write each exponential expression as the power of the same base, we will consider taking logarithm of both sides of the equation. While any appropriate base can be used, the best practical base is base 10 or base e. It is easier if the exponential expression is isolated on one side of the equation.

Example 10 Solve an Exponential Equation

Solve: $3^x = 7$.
Solution
Take base 10 logarithm of both sides, to get
$$\log 3^x = \log 7$$
Rewrite the equation using the properties of logarithms
$$x \log 3 = \log 7$$
Solve for x.
$$x = \frac{\log 7}{\log 3} \approx 1.771$$
The solution rounded to three decimal places is 1.771.

Example 11 **Solve an Exponential Equation**

Solve: $2^{x-1} = 5^{3x-5}$

Solution

Take base 10 logarithm of both sides, to get

$$\log 2^{x-1} = \log 5^{3x-5}$$

Rewrite the equation using the power property of logarithms.

$$(x-1)\log 2 = (3x-5)\log 5$$

$$x\log 2 - \log 2 = 3x\log 5 - 5\log 5$$

Collect terms that involve the variable x on the left side.

$$x\log 2 - 3x\log 5 = \log 2 - 5\log 5$$
$$x(\log 2 - 3\log 5) = \log 2 - 5\log 5$$

$$x = \frac{\log 2 - 5\log 5}{\log 2 - 3\log 5} \approx 1.78$$

The solution rounded to two decimal places is 1.78.

Logarithmic Equations

Equations that contain logarithms are called **logarithmic equations.** The properties of logarithms, together with the definition of a logarithm, are used to solve logarithmic equations. The following steps can be valuable aids in solving logarithmic equations.

1. Use the product and quotient properties of logarithms to get a single logarithm on each side of the equation.

2. Use one of the following properties to eliminate logarithms.

If $\log_b x = \log_b y$ then $x = y$

If $\log_b x = k$, then $x = b^k$

3. Solve for the variable.

4. Check solutions.

Caution: Be sure to check all solutions. Solving a logarithmic equation by using logarithmic properties may introduce extraneous solutions.

Example 12 **Solve a Logarithmic Equation**

Solve: $\log_4(x-1) = 2$.

Solution

Rewrite the equation in exponential form, and solve for x.

$$x - 1 = 4^2$$
$$x - 1 = 16$$
$$x = 17$$

To check the possible solution, replace x by 17 in the original equation. 17 checks as a solution.

Example 13 **Solve a Logarithmic Equation**

Solve: $\log_2(6x+2) - \log_2(3-x) = 1$.

Solution

$\log_2(6x+2) - \log_2(3-x) = 1$

$\log_2 \dfrac{6x+2}{3-x} = 1$ • Quotient property.

$\dfrac{6x+2}{3-x} = 2^1$ • Power property.

$6x+2 = 2(3-x)$
$6x+2 = 6-2x$
$8x = 4$
$x = \frac{1}{2}$

It can be shown that $\frac{1}{2}$ checks as a solution of the equation.

Example 14 **Solve a Logarithmic Equation**

Solve: $\log_7 x = \log_7 2 + \log_7(x^2-5)$

Solution

$\log_7 x = \log_7 2 + \log_7(x^2-5)$
$\log_7 x = \log_7 2(x^2-5)$
$\log_7 x = \log_7(2x^2-10)$
$x = 2x^2 - 10$
$2x^2 - x - 10 = 0$
$(2x-5)(x+2) = 0$
$2x-5=0$ or $x+2=0$
$x = \frac{5}{2}$ $\quad x=-2$

The possible solutions are $\frac{5}{2}$ and -2.
Now, check these solutions in the original problem.

Check $x = \frac{5}{2}$

$\log_7 x = \log_7 2 + \log_7(x^2-5)$?

$\log_7 \frac{5}{2} = \log_7 2 + \log_7\left[\left(\frac{5}{2}\right)^2 - 5\right]$?

$\log_7 \frac{5}{2} = \log_7 2 + \log_7 \frac{5}{4}$ **True**

Using a calculator we can verify that the above equation is true, and $\frac{5}{2}$ is a solution of the equation.

Check $x = -2$

$\log_7(-2) = \log_7 2 + \log_7\left[(-2)^2 - 5\right]$?

$\log_7(-2) = \log_7 2 + \log_7(-1)$ **False**

Since $\log_7(-2)$ and $\log_7(-1)$ are not defined, -2 does not check as a solution. Therefore, the solution set is $\left\{\frac{5}{2}\right\}$.

7.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Compound Interest Models

The model used for compound interest paid annually is

$$A = P(1+r)^t$$

The model used for compound interest paid more often than yearly is

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

For quarterly compounding use $n = 4$
For monthly compounding use $n = 12$
For daily compounding use $n = 365$

The model used for interest compounded continuously is

$$A = Pe^{rt}$$

where,

$A =$ The value of the investment in future
$P =$ The principal
$r =$ Interest rate per year
$t =$ Number of years
$n =$ Number of compounding periods per year.

Example 15 **Solve a Compound Interest Application**

Suppose \$2500 is deposited in an account at an annual interest rate of 5% compounded quarterly. How much will be in the account after 8 years?

Solution

Use $A = P\left(1+\frac{r}{n}\right)^{nt}$, with $P = 2500,\ r = 0.05,\ t = 8,\ n = 4.$

Then,

$$A = 2500\left(1+\frac{0.05}{4}\right)^{4\cdot 8}$$

$$A = 2500(1+0.0125)^{32}$$
$$A = 2500(1.0125)^{32}$$
$$A \approx 3,720.33$$

The amount in the account rounded to two decimal places is \$3,720.33.

Example 16 **Tripling an Investment**

How long will it take \$8,000 to triple if it is invested in an account that pays 6% annual interest compounded continuously?

Solution

Use $A = Pe^{rt}$, with $A = 24000,\ P = 8000,\ r = 0.06$

Then,

$$24,000 = 8,000e^{0.06t}$$

$$3 = e^{0.06t}$$ • Divide each side by 8,000.

$$\ln 3 = 0.06t$$ • Take the natural logarithm of each side.

$$t = \frac{\ln 3}{0.06}$$ • Solve for t.

$$t \approx 18$$

Therefore, it takes approximately 18 years for the investment to triple.

Exponential Growth

A model for population or substance growth is given by

$$P = P_0 e^{rt}, \qquad r > 0$$

where,

$P =$ Population at time t (future population.)
$P_0 =$ Population at time $t = 0$ (initial population.)
$r =$ Growth rate
$t =$ Time unit

Example 17 **Bacterial Growth**

A population of a colony of bacteria increases according to the growth model $P = P_0 e^{0.12t}$ (where t is measured in days). How long will it take for the population of $3,000$ to double?

Solution

Use $P = P_0 e^{0.12t}$ with $P = 6000$ and $P_0 = 3000$.

$$6,000 = 3,000e^{0.12t}$$

$$2 = e^{0.12t}$$ • Divide each side by 3,000.

$$\ln 2 = 0.12t$$ • Take the natural logarithm of each side.

$$t = \frac{\ln 2}{0.12} \approx 6$$

The population will double in approximately 6 days.

Exponential Decay

A model for population or substance decay is given by

$$P = P_0 e^{rt}, \qquad r < 0$$

where,

$P =$ Population at time t.
$P_0 =$ Population at time $t = 0$.
$r =$ Decay rate
$t =$ Time unit

Remark: *The half-life of a substance, is the amount of time it takes for half of the initial amount to decay.*

Example 18 **Estimating Half-life of a substance**

A certain radioactive substance decays according to the law $P = P_0 e^{-0.03t}$, where P_0 is the initial amount present and P is the amount present at time t (in hours). What is the half-life of this substance?

Solution

Replace P with $\frac{1}{2}P_0$ in $P = P_0 e^{-0.03t}$, and then solve for t.

$$\tfrac{1}{2}P_0 = P_0 e^{-0.03t}$$

$$\tfrac{1}{2} = e^{-0.03t}$$ • Divide each side by P_0.

$$\ln \tfrac{1}{2} = -0.03t$$ • Take natural logarithm of each side.

$$t = \frac{\ln \frac{1}{2}}{-0.03} \approx 23$$ • Solve for t.

Therefore, the half-life of this substance is about 23 hours.

Newton's Law of Cooling

A model for cooling a substance is given by

$$u = T + (u_0 - T)\,e^{kt}\ ,\ k < 0$$

where,

$u =$ Temperature of a heated object at time t.
$u_0 =$ Initial temperature of the heated object
$T =$ Constant temperature of the surrounding
$k =$ Constant negative number

Example 19 **Using Newton's Law of Cooling**

Suppose a bowl of soup at temperature $85°C$ is placed in a room with temperature of $25°C$. If the temperature of the bowl of soup is $60°C$ after 5 minutes, when will the temperature of the bowl be $35°$?

Solution

Use the equation

$$u = T + (u_0 - T)\,e^{kt} \qquad (1)$$

with $u_0 = 85,\ T = 25,\ u = 60$, and $t = 5$.

$$60 = 25 + (85 - 25)\,e^{5k}$$
$$35 = 60e^{5k}$$
$$\frac{35}{60} = e^{5k}$$

To find k take natural logarithm of each side of the equation to get,

$$\ln\frac{35}{60} = 5k$$

Solve for k.

$$k = \frac{\ln\frac{35}{60}}{5} \approx -0.1078$$

Therefore, formula (1) becomes

$$u = 25 + (85 - 25)\,e^{-0.1078t}$$
$$u = 25 + 60e^{-0.1078t} \qquad (2)$$

Now we want to find t when $u = 35°C$, to do this substitute 35 for u in equation (2).

$$35 = 25 + 60e^{-0.1078t}$$
$$10 = 60e^{-0.1078t}$$
$$\frac{1}{6} = e^{-0.1078t}$$

Take natural logarithm of each side.

$$\ln\tfrac{1}{6} = -0.1078t$$
$$t = \frac{\ln\frac{1}{6}}{-0.1078} \approx 16.6 \text{ minutes}$$

Therefore, the temperature of the bowl will be $35°C$ after about 16.6 minutes.

Loudness of Sound

A model for loudness of sound is

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where,

$L =$ Loudness of sound measured in decibels

$I =$ Intensity of sound measured in watts per square centimeter

$I_0 =$ Intensity of the lowest sound that can be heard, internationally agreed upon to be 10^{-16}

Example 20 **Sound Intensity**

The intensity of light rainfall is 10^{-14} watts per square centimeters. What is the loudness of light rainfall in decibels?

Solution

Use $L = 10\log\left(\frac{I}{I_0}\right)$ with $I = 10^{-14}$ and $I_0 = 10^{-16}$.

$$L = 10\log\left(\frac{10^{-14}}{10^{-16}}\right)$$

$$L = 10\log 10^2$$

$$L = 20 \quad \text{decibels}$$

The loudness of light rainfall is 20 decibels.

SOLVED PROBLEMS

Graphs of Exponential Functions

Sketch the graph of each function using the graphing techniques.

1. $f(x) = 1 - 4^{-x}$

Solution

Use the following steps to graph $f(x) = 1 - 4^{-x}$.

Step 1: $y = 4^{-x}$ *Reflect about y-axis*

Step 2: $y = -4^{-x}$ *Reflect about x-axis*

Step 3: $y = 1 - 4^{-x}$ *Vertical shift up 1 unit*

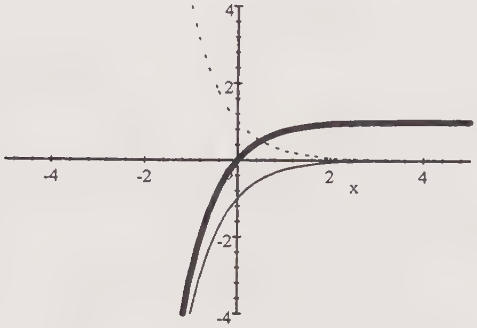


Figure 7.8

2. $f(x) = 4^{x-2}$

Solution

Use the following steps to graph $f(x) = 4^{x-2}$.

Step 1: $y = 4^{x}$ *Exponential function*

Step 2: $y = 4^{x-2}$ *Horizontal shift right 2 units. Horizontal shifts will not change the asymptote. The y-intercept is* $\frac{1}{16}$.

Note: $y - intercept = 4^{-2} = \frac{1}{16}$

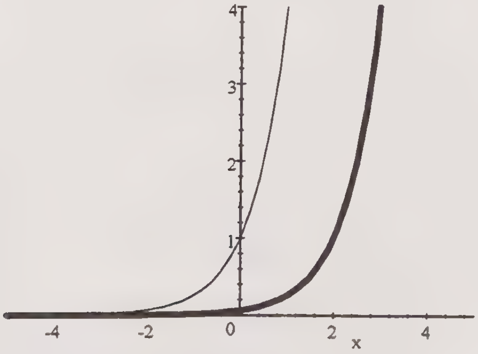


Figure 7.9

Evaluating Logarithmic Functions

Use equality of exponents property to evaluate each logarithm.

3. $\log_5 125$

Solution

Let $\log_5 125 = x$. Then $5^x = 125$, so

$$5^x = 125$$
$$5^x = 5^3$$

$$x = 3$$

Therefore,

$$\log_5 125 = 3$$

4. $\log_{16}\left(\frac{1}{2}\right)$

Solution

Let $\log_{16}\left(\frac{1}{2}\right) = x$. Then $16^x = \frac{1}{2}$, so

$$16^x = \tfrac{1}{2}$$

$$\left(2^4\right)^x = 2^{-1}$$

$$2^{4x} = 2^{-1}$$

$$4x = -1 \qquad \bullet \text{ Equate exponents.}$$

$$x = -\tfrac{1}{4}$$

Therefore,

$$\log_{16}\left(\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

5. $\log(0.0001)$

Solution

Let $\log(0.0001) = x$. Then $10^x = 0.0001$, so

$$10^x = 0.0001$$

$$10^x = 10^{-4}$$

$$x = -4$$

Therefore,

$$\log(0.0001) = -4$$

Use the change-of-base formula to approximate each logarithm to three decimal places.

6. $\log_3 7$

Solution

$$\log_3 7 = \frac{\log 7}{\log 3} \approx 1.771$$

7. $\log_{\sqrt{2}}\left(\sqrt{7}\right)$

Solution

$$\log_{\sqrt{2}}\left(\sqrt{7}\right) = \frac{\log\sqrt{7}}{\log\sqrt{2}} \approx 2.807$$

Graphs of Logarithmic Functions

Sketch the graph of each function using the graphing techniques.

8. $f(x) = -\log_2 x + 3$

Solution

Use the following steps to graph $f(x) = -\log_2 x + 3$.

Step 1: $y = \log_2 x$ *Logarithmic function*

Step 2: $y = -\log_2 x$ *Reflect about x-axis*

Step 3: $y = -\log_2 x + 3$ *Vertical shift up 3 units. Vertical shifts will not change the asymptote. The x-intercept is:*

$$\log_2 x = 3 \implies x = 2^3 = 8$$

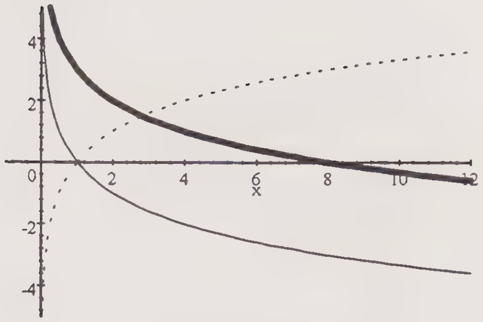


Figure 7.10

9. $f(x) = \log(3 - x)$

Solution

Use the following steps to get the graph of $y = \log(3 - x)$.

Step 1: $y = \log x$ *Logarithmic function*

Step 2: $y = \log(3 + x)$ *Horizontal shift 3 units to the left*

Step 3: $y = \log(3 - x)$ *Reflect about y-axis*

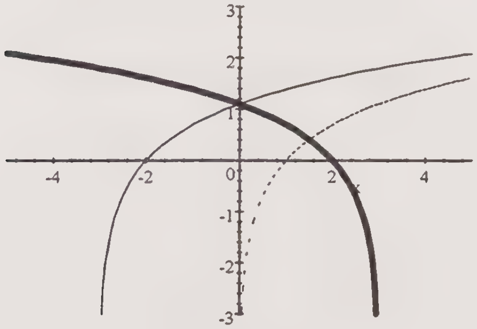


Figure 7.11

Domain of Logarithmic Functions

Find the domain of each logarithmic function.

10. $f(x) = \log\left(x^2 - x - 20\right)$

Solution

Domain of f consists of all x such that

$$x^2 - x - 20 > 0$$

To solve the quadratic inequality, first factor the expression then set each factor equal to zero.

$$(x-5)(x+4) > 0$$
$$x - 5 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 5 \qquad\qquad x = -4$$

	$-\infty < x < -4$	$-4 < x < 5$	$5 < x < \infty$
x	-6	-2	6
$x - 5$	$-$	$-$	$+$
$x + 4$	$-$	$+$	$+$
$x^2 - x - 20 > 0$	$+$	$-$	$+$

Domain $= \{x \mid -\infty < x < -4 \quad \text{or} \quad 5 < x < \infty\}$.

11. $h(x) = \log_3 |x+2|$

Solution

The domain of h consists of all x such that

$$|x+2| > 0$$

Domain $= \{x \mid x > -2 \quad \text{or} \quad x < -2\}$.

12. $g(x) = \log_5 \left(\frac{2}{3-x}\right)$

Solution

The domain of g consists of all x for which $\frac{2}{3-x} > 0$

Set the denominator equal to zero, to get

$$3 - x = 0$$
$$x = 3$$

	$-\infty < x < 3$	$3 < x < \infty$
x	2	4
$3 - x$	+	−
$\frac{2}{3-x} > 0$	+	−

Domain $= \{x \mid -\infty < x < 3\}$.

Properties of Logarithms

Write each expression in the expanded form.

13. $\log_3 (4x)$

Solution

$$\log_3 (4x) = \log_3 4 + \log_3 x$$

14. $\log_4 x^3 y^2$

Solution

$$\log_4 x^3y^2 = \log_4 x^3 + \log_4 y^2$$
$$= 3\log_4 x + 2\log_4 y$$

15. $\log_2 a^3 b^4 c^5$

Solution

$$\log_2 a^3b^4c^5 = \log_2 a^3 + \log_2 b^4 + \log_2 c^5$$
$$= 3\log_2 a + 4\log_2 b + 5\log_2 c$$

16. $\log_3 \sqrt{xy^3}$

Solution

$$\log_3 \sqrt{xy^3} = \log_3 (xy^3)^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\log_3 (xy^3)$$

$$= \tfrac{1}{2}(\log_3 x + \log_3 y^3)$$

$$= \tfrac{1}{2}(\log_3 x + 3\log_3 y)$$

$$= \tfrac{1}{2}\log_3 x + \tfrac{3}{2}\log_3 y$$

17. $\log_2\left(\frac{xy^3}{2}\right)$

Solution

$$\begin{aligned}\log_2\left(\frac{xy^3}{2}\right) &= \log_2 xy^3 - \log_2 2\\ &= \log_2 x + \log_2 y^3 - 1 && \bullet \log_2 2 = 1.\\ &= \log_2 x + 3\log_2 y - 1\end{aligned}$$

18. $\log_8 (2ab^2)^3$

Solution

$$\begin{aligned}\log_8 (2ab^2)^3 &= \log_8 (8a^3b^6)\\ &= \log_8 8 + \log_8 a^3 + \log_8 b^6 && \bullet \log_8 8 = 1.\\ &= 1 + 3\log_8 a + 6\log_8 b\end{aligned}$$

19. $\log_2 \sqrt[3]{\frac{x}{y}}$

Solution

$$\begin{aligned}\log_2 \sqrt[3]{\frac{x}{y}} &= \log_2\left(\frac{x}{y}\right)^{\frac{1}{3}}\\ &= \tfrac{1}{3}\log_2\left(\frac{x}{y}\right)\\ &= \tfrac{1}{3}(\log_2 x - \log_2 y)\\ &= \tfrac{1}{3}\log_2 x - \tfrac{1}{3}\log_2 y\end{aligned}$$

20. $\log_2 \sqrt[4]{\frac{x^3y^2}{16}}$

Solution

$$\begin{aligned}\log_2 \sqrt[4]{\frac{x^3y^2}{16}} &= \log_2\left(\frac{x^3y^2}{16}\right)^{\frac{1}{4}}\\ &= \tfrac{1}{4}\log_2\left(\frac{x^3y^2}{16}\right)\\ &= \tfrac{1}{4}\left(\log_2 x^3y^2 - \log_2 16\right)\\ &= \tfrac{1}{4}\left(\log_2 x^3 + \log_2 y^2 - \log_2 16\right)\\ &= \tfrac{1}{4}\left(3\log_2 x + 2\log_2 y - \log_2 2^4\right)\\ &= \tfrac{1}{4}(3\log_2 x + 2\log_2 y - 4\log_2 2) && \bullet \log_2 2 = 1.\\ &= \tfrac{1}{4}(3\log_2 x + 2\log_2 y - 4)\\ &= \tfrac{3}{4}\log_2 x + \tfrac{1}{2}\log_2 y - 1\end{aligned}$$

21. $\ln\sqrt{\frac{5}{e^2}}$

Solution

$$\ln\sqrt{\frac{5}{e^2}} = \ln\left(\frac{5}{e^2}\right)^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\ln\left(\frac{5}{e^2}\right)$$

$$= \tfrac{1}{2}\left(\ln 5 - \ln e^2\right)$$

$$= \tfrac{1}{2}(\ln 5 - 2\ln e)$$

$$= \tfrac{1}{2}(\ln 5 - 2) \qquad \bullet\ \ln e = \log_e e = 1.$$

$$= \tfrac{1}{2}\ln 5 - 1$$

Write each expression as a single logarithm with coefficient of one.

22. $\log_2 x^3y^2 - \log_2 xy^2$

Solution

$$\log_2 x^3y^2 - \log_2 xy^2 = \log_2\left(\frac{x^3y^2}{xy^2}\right)$$

$$= \log_2 x^2$$

23. $3\log_5 x + 2\log_5 y$

Solution

$$3\log_5 x + 2\log_5 y = \log_5 x^3 + \log_5 y^2$$
$$= \log_5 x^3y^2$$

24. $-2\log_5 x + 4\log_5 x$

Solution

$$-2\log_5 x + 4\log_5 x = \log_5 x^{-2} + \log_5 x^4$$

$$= \log_5\left(x^{-2}\cdot x^4\right)$$

$$= \log_5 x^2$$

25. $\log_2\left(x^2 - 25\right) - \log_2(x+5)$

Solution

$$\log_2\left(x^2 - 25\right) - \log_2(x+5) = \log_2\left(\frac{x^2-25}{x+5}\right)$$

$$= \log_2\frac{(x+5)(x-5)}{(x+5)}$$

$$= \log_2(x-5)$$

26. $2\log x + 3\log y - 2\log xy$

Solution

$$2\log x + 3\log y - 2\log xy = \log x^2 + \log y^3 - \log(xy)^2$$

$$= \log x^2y^3 - \log x^2y^2$$

$$= \log\left(\frac{x^2y^3}{x^2y^2}\right)$$

$$= \log y$$

27. $\tfrac{1}{2}(\log_3 a - 2\log_3 b + \log_3 c)$

Solution

$$\tfrac{1}{2}(\log_3 a - 2\log_3 b + \log_3 c) = \tfrac{1}{2}\log_3\left(\frac{ac}{b^2}\right)$$

$$= \log_3 \left(\frac{ac}{b^2}\right)^{\frac{1}{2}}$$

28. $\frac{1}{3}\log_3 x - \frac{1}{3}\log_3 y$

Solution

$$\begin{aligned}\tfrac{1}{3}\log_3 x - \tfrac{1}{3}\log_3 y &= \tfrac{1}{3}\left(\log_3 x - \log_3 y\right)\\ &= \tfrac{1}{3}\log_3\left(\frac{x}{y}\right)\\ &= \log_3\left(\frac{x}{y}\right)^{\frac{1}{3}}\end{aligned}$$

29. $\frac{1}{4}\log_2 a^4 + \frac{1}{4}\log_2 64 - \frac{1}{4}\log_2 4$

Solution

$$\begin{aligned}\tfrac{1}{4}\log_2 a^4 + \tfrac{1}{4}\log_2 64 - \tfrac{1}{4}\log_2 4 &= \tfrac{1}{4}\left(\log_2 a^4 + \log_2 64 - \log_2 4\right)\\ &= \tfrac{1}{4}\log_2\left(\tfrac{64a^4}{4}\right)\\ &= \tfrac{1}{4}\log_2\left(16a^4\right)\\ &= \log_2\left(16a^4\right)^{\frac{1}{4}}\\ &= \log_2 \sqrt[4]{16a^4}\\ &= \log_2 2a\end{aligned}$$

Exponential Equations

Solve each exponential equation.

30. $2^x = \frac{1}{8}$

Solution

$$2^x = \frac{1}{8}$$

$2^x = 2^{-3}$ • Write each side as a power of base 2.

$x = -3$ • Equate the exponents.

The solution set is $\{-3\}$.

31. $25^x = 125$

Solution

$$\begin{aligned}25^x &= 125\\ \left(5^2\right)^x &= 5^3\\ 5^{2x} &= 5^3\\ 2x &= 3\\ x &= \tfrac{3}{2}\end{aligned}$$

The solution set is $\left\{\frac{3}{2}\right\}$.

32. $3^{x^2-15} = 9^x$

Solution

$$\begin{aligned}3^{x^2-15} &= 9^x\\ 3^{x^2-15} &= 3^{2x}\\ x^2 - 15 &= 2x\end{aligned}$$

$x^2 - 2x - 15 = 0$
$(x+3)(x-5) = 0$
$x + 3 = 0$ *or* $x - 5 = 0$
$x = -3 \qquad x = 5$
The solution set is $\{-3, 5\}$.

33. $9^{x^2} \cdot 3^x = 27$
Solution

$$9^{x^2} \cdot 3^x = 27$$

$$\left(3^2\right)^{x^2} \cdot 3^x = 3^3$$

$$3^{2x^2} \cdot 3^x = 3^3$$

$$3^{2x^2+x} = 3^3$$

$$2x^2 + x = 3$$
$$2x^2 + x - 3 = 0$$
$$(2x+3)(x-1) = 0$$
$2x + 3 = 0$ or $x - 1 = 0$
$x = -\frac{3}{2} \qquad x = 1$
The solution set is $\left\{1, -\frac{3}{2}\right\}$.

34. $(7)^{2x} \cdot (7)^{4x} = 49$
Solution

$$7^{2x} \cdot 7^{4x} = 49$$

$$7^{2x} \cdot 7^{4x} = 7^2$$

$$7^{2x+4x} = 7^2$$

$$2x + 4x = 2$$
$$6x = 2$$
$$x = \frac{1}{3}$$
The solution set is $\left\{\frac{1}{3}\right\}$.

35. $5^x = 6$
Solution

$5^x = 6$
$\log 5^x = \log 6$ • Take the common logarithm of each side.
$x \log 5 = \log 6$ • Rewrite the equation, using properties of logarithms.

$x = \dfrac{\log 6}{\log 5}$ • Solve for x.

$x \approx 1.113$ • Use a calculator.

36. $2^{3x} = 36$
Solution

$2^{3x} = 36$
$\log 2^{3x} = \log 36$ • Take the common logarithm of each side.
$(3x) \log 2 = \log 36$ • Use power property.

$x = \dfrac{\log 36}{3 \log 2}$ • Solve for x.

$x \approx 1.723$ • Use a calculator.

37. $2^{3x-1} = 7^{1-x}$

Solution

$2^{3x-1} = 7^{1-x}$

$\log 2^{3x-1} = \log 7^{1-x}$ • Take the common logarithm of both sides.

$(3x-1)\log 2 = (1-x)\log 7$ • Power property

$3x\log 2 - \log 2 = \log 7 - x\log 7$ • Distributive property

$3x\log 2 + x\log 7 = \log 2 + \log 7$

$x(3\log 2 + \log 7) = \log 2 + \log 7$ • Collect terms.

$x = \dfrac{\log 2 + \log 7}{3\log 2 + \log 7}$ • Solve for x.

$x \approx 0.66$ • Use a calculator.

38. $2^{3x+1} = 6$

Solution

$2^{3x+1} = 6$

$\log 2^{3x+1} = \log 6$ • Take the common logarithm of each side.

$(3x+1)\log 2 = \log 6$ • Apply power property.

$3x\log 2 + \log 2 = \log 6$ • Apply distributive property.

$3x\log 2 = \log 6 - \log 2$ • Collect like terms.

$$x = \frac{\log 6 - \log 2}{3\log 2} \approx 0.528$$

39. $e^{2x} = 300$

Solution

$e^{2x} = 300$

$\ln e^{2x} = \ln 300$ • Take the natural logarithm of each side.

$2x = \ln 300$ • Use power property, and $\ln e = 1$.

$x = \dfrac{\ln 300}{2}$ • Solve for x.

$x \approx 2.852$

40. $\left(\frac{1}{4}\right)^{2x} = 5$

Solution

$$\left(\frac{1}{4}\right)^{2x} = 5$$

$$\log\left(\frac{1}{4}\right)^{2x} = \log 5$$

$$2x\log\left(\frac{1}{4}\right) = \log 5$$

$$x = \frac{\log 5}{2\log\left(\frac{1}{4}\right)}$$

$$x \approx -0.5805$$

Logarithmic Equations

Solve each logarithmic equation.

41. $\log_5 x = 2$

Solution

$$\log_5 x = 2$$
$$x = 5^2$$
$$x = 25$$

The solution set is $\{25\}$.

42. $\log_{\frac{1}{2}} 8 = x$

Solution

$$\log_{\frac{1}{2}} 8 = x$$

$$8 = \left(\frac{1}{2}\right)^x$$

$$2^3 = 2^{-x}$$
$$3 = -x \quad \text{so} \quad x = -3$$

The solution set is $\{-3\}$.

43. $\log_3 x = 0$

Solution

$$\log_3 x = 0$$
$$x = 3^0 = 1$$

The solution set is $\{1\}$.

44. $\log_2 x = -4$

Solution

$$\log_2 x = -4$$
$$x = 2^{-4}$$

$$x = \frac{1}{2^4} = \frac{1}{16}$$

The solution set is $\{\frac{1}{16}\}$.

45. $\log_2 8 = x^2 - 6$

Solution

$$\log_2 8 = x^2 - 6$$

$$8 = 2^{x^2-6}$$

- Write equation in exponential form.

$$2^3 = 2^{x^2-6}$$

- Write each side as a power of 2.

$$3 = x^2 - 6$$

- Equate exponents.

$$-x^2 = -9$$
$$x^2 = 9$$
$$x = \pm 3$$

The solution set is $\{-3, +3\}$.

46. $\log_5 \frac{1}{25} = 2x^2 - 4$

Solution

$$\log_5 \frac{1}{25} = 2x^2 - 4$$

$$\frac{1}{25} = 5^{2x^2-4}$$

- Write equation in exponential form.

$5^{-2} = 5^{2x^2-4}$ • Write each side as a power of 5.
$-2 = 2x^2 - 4$ • Equate exponents.
$-2x^2 = -2$ • Combine like terms.
$x^2 = 1$
$x = \pm 1$
The solution set is $\{-1, +1\}$.

47. $\log_2 9x - \log_2 (x^2 - 1) = \log_2 6$
Solution
$\log_2 9x - \log_2 (x^2 - 1) = \log_2 6$

$\log_2 \left(\frac{9x}{x^2-1}\right) = \log_2 6$ • Quotient property.

$\frac{9x}{x^2-1} = 6$ • Use $\log_a x = \log_a y$ then $x = y$..

$$\begin{aligned} 6x^2 - 6 &= 9x \\ 6x^2 - 9x - 6 &= 0 \\ 2x^2 - 3x - 2 &= 0 \\ (2x+1)(x-2) &= 0 \end{aligned}$$
$2x + 1 = 0$ or $x - 2 = 0$
$x = -\frac{1}{2}$ $x = 2$
Thus, $-\frac{1}{2}$ and 2 are possible solutions.
Now, check the possible solutions in the original equation.

Check $x = -\frac{1}{2}$
$\log_2 \left[9\left(-\frac{1}{2}\right)\right] - \log_2 \left[\left(-\frac{1}{2}\right)^2 - 1\right] = \log_2 6$?

$\log_2 \left(-\frac{9}{2}\right) - \log_2 \left(-\frac{3}{4}\right) = \log_2 6$ **False**

Since the logarithm of a negative number is not defined, $-\frac{1}{2}$ does not check as a solution.

Check $x = 2$
$\log_2 9(2) - \log_2 \left[(2)^2 - 1\right] = \log_2 6$?
$\log_2 18 - \log_2 3 = \log_2 6$ **True**

We can use a calculator to verify that the above statement is true and that 2 checks as a solution of the equation. Therefore, the solution set is $\{2\}$.

48. $\log_4 x + \log_4 (4 - x) = \log_4 3$
Solution
$\log_4 x(4 - x) = \log_4 3$ • Product property.
$x(4 - x) = 3$ • Use $\log_a x = \log_a y$ then $x = y$.
$4x - x^2 - 3 = 0$
$x^2 - 4x + 3 = 0$
$(x-3)(x-1) = 0$
$x - 3 = 0$ or $x - 1 = 0$
$x = 3$ $x = 1$
The possible solutions are 3 and 1.
Now, check the possible solutions in the original equation.

Check $x = 3$

$\log_4 \mathbf{3} + \log_4 (4 - \mathbf{3}) = \log_4 3$?

$\log_4 3 + \log_4 1 = \log_4 3$?

$\log_4 3 = \log_4 3$ **True** • $\log_4 1 = 0$

Check $x = 1$

$\log_4 \mathbf{1} + \log_4 (4 - \mathbf{1}) = \log_4 3$?

$\log_4 1 + \log_4 3 = \log_4 3$?

$\log_4 3 = \log_4 3$ **True** • $\log_4 1 = 0$

Therefore, 3 and 1 check as solutions of the equation. Therefore, the solution set is $\{1, 3\}$.

49. $\log_3 (2x - 3) = 3$

Solution

$\log_3 (2x - 3) = 3$

$2x - 3 = 3^3$

$2x - 3 = 27$

$2x = 30$

$x = 15$

Check $x = 15$

$\log_3 (2x - 3) = 3$

$\log_3 [2(\mathbf{15}) - 3] = 3$?

$\log_3 (30 - 3) = 3$?

$\log_3 27 = 3$ **True**

Using a calculator, we can verify that 15 checks as a solution of the equation. Therefore, the solution set is $\{15\}$.

50. $\log_4 (x^2 + 6x) = 2$

Solution

$\log_4 (x^2 + 6x) = 2$

$x^2 + 6x = 4^2$

$x^2 + 6x = 16$

$x^2 + 6x - 16 = 0$

$(x - 2)(x + 8) = 0$

$x - 2 = 0$ or $x + 8 = 0$

$x = 2$ $x = -8$

Now, check the proposed solutions.

Check $x = 2$

$\log_4 (x^2 + 6x) = 2$

$\log_4 \left[(\mathbf{2})^2 + 6(\mathbf{2})\right] = 2$?

$\log_4 (4 + 12) = 2$?

$\log_4 16 = 2$ **True**

Check $x = -8$

$\log_4 (x^2 + 6x) = 2$

$\log_4 \left[(\mathbf{-8})^2 + 6(\mathbf{-8})\right] = 2$?

$\log_4 (64 - 48) = 2$?

$\log_4 16 = 2$ **True**

The proposed solutions check in the original equation. Therefore, the solution set is $\{2, -8\}$.

Applications

51. An investor deposits $\$15,000$ in an account that earns 9% interest per year compounded annually, what is the value of the investment after 4 years?

Solution

Use $A = P(1+r)^t$, with $P = 15,000$, $r = 0.09$, and $t = 4$.

$$\begin{aligned} A &= P(1+r)^t \\ A &= 15,000(1+0.09)^4 \\ A &= 15,000(1.09)^4 \\ A &= 21,173.72 \end{aligned}$$

Therefore, the $\$15,000$ investment will be worth $\$21,173.72$ at the end of the 4 years.

52. How long will it take for an investment of $\$2,000$ to grow to $\$8,000$ at 5.25% per annum interest compounded annually?

Solution

Use $A = P(1+r)^t$ with $P = 2,000$, $A = 8,000$, and $r = 0.0525$.

$$\begin{aligned} A &= P(1+r)^t \\ 8,000 &= 2,000(1+0.0525)^t \\ 4 &= (1.0525)^t \qquad \bullet \text{ Divide each side by 2,000.} \\ \log 4 &= \log(1.0525)^t \\ \log 4 &= t\log(1.0525) \end{aligned}$$

$$t = \frac{\log 4}{\log(1.0525)} \approx 27.092$$

Thus, it will take approximately 27 years.

53. For a $\$10,000$ investment in a bond trust that pays interest of 10% per year compounded quarterly, what will be the value of investment after 6 years?

Solution

Use the equation $A = P\left(1+\frac{r}{n}\right)^{nt}$, with $P = 10000$, $r = 0.10$, $t = 6$ and $n = 4$.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 10,000\left(1+\frac{0.10}{4}\right)^{4(6)}$$

$$A = 10,000(1+0.025)^{24}$$

$$A = 10,000(1.025)^{24}$$

$$A \approx 18,087.26$$

Therefore, the value of $\$10,000$ after 6 years will be approximately $\$18,087.26$.

54. What is the value of an investment of $\$600$ at an interest rate of 8% per year compounded continuously after $4\frac{1}{2}$ years?

Solution

Use the equation $A = Pe^{rt}$, with $P = 600$, $r = 0.08$, $t = 4\frac{1}{2}$.

$$\begin{aligned} A &= Pe^{rt} \\ A &= 600e^{0.08(4.5)} \\ A &\approx 860 \end{aligned}$$

Therefore, the value of investment after $4\frac{1}{2}$ years will be $\$860$.

55. How much interest would Linda earn by depositing \$5, 400 for 3 years at an interest rate of $5\frac{1}{2}\%$ per year compounded annually?

Solution

Use the equation $A = P(1+r)^t$, with $P = 5400$, $r = 0.055$ and $t = 3$ to find the value of money after 3 years.

$$A = P(1+r)^t$$
$$A = 5400(1+0.055)^3$$
$$A = 6,340.90$$
$$\text{Interest} = A - P$$
$$\text{Interest} = 6340.90 - 5400 = \$940.90.$$

56. If the population of a town is 10,000 people, how many years will it take for the population to double? Assume a continuous growth rate of 2% per year.

Solution

Use the equation $P = P_0e^{rt}$, with $P = 20,000$, $P_0 = 10,000$, $r = 0.02$.

$$P = P_0e^{rt}$$
$$20,000 = 10,000e^{0.02t}$$
$$2 = e^{0.02t}$$
$$\ln 2 = 0.02t$$

• Take the natural logarithm of each side.

$$t = \frac{\ln 2}{0.02}$$
$$t \approx 34.66$$

Therefore, it will take approximately 35 years for the population of town to double.

57. A town in north Peru has a population of 1.3 million people. How long ago was the population 540, 000? (Assume a continuous growth rate of 2%)

Solution

Use the population equation $P = P_0e^{rt}$ with $P = 1.3$ million, $P_0 = 540,000$ and $r = 0.02$.

$$P = P_0e^{rt}$$
$$1,300,000 = 540,000e^{0.02t}$$
$$2.407 = e^{0.02t}$$

• Divide each side by 540,000.

$$\ln 2.407 = 0.02t$$
$$t = \frac{\ln 2.407}{0.02} \approx 44$$

Therefore, approximately 44 years ago population of the town was 540,000.

58. Strontium-90 is a radioactive material that has a half-life of 28 years. How long does it take for 750 grams of Strontium-90 to decay to 25 grams?

Solution

Replace P with $\frac{1}{2}P_0$ and t with 28 in the decay function $P = P_0e^{rt}$, and solve for r.

$$P = P_0e^{rt}$$
$$\tfrac{1}{2}P_0 = P_0e^{r(28)}$$

$$\frac{1}{2} = e^{28r} \qquad \bullet \text{ Divide each side by } P_0.$$

$$\ln \tfrac{1}{2} = 28r$$

$$r = \frac{\ln \frac{1}{2}}{28} \approx -0.02476$$

Therefore, $P = P_0 e^{-0.02476t}$ (1)

To find the time required for the 750 grams of Strontium-90 to decay to 25 grams, let $P_0 = 750$, $P = 25$ in equation (1), and solve for t.

$$25 = 750e^{-0.02476t}$$

$$\frac{25}{750} = e^{-0.02476t}$$

$$\ln \frac{25}{750} = -0.0247t$$

$$t = \frac{\ln \frac{25}{750}}{-0.02476} \approx 137$$

Therefore, it will take approximately 137 years for 750 grams of Stronium-90 to decay to 25 grams.

59. An object is heated to $100°C$ and then placed in a refrigerator with temperature of $5°C$. The temperature (in degrees Celsius) of the object after t minutes is given by $u = 5 + 95e^{-0.15t}$. Find the temperature of the object after one hour.

Solution

$$u = 5 + 95e^{-0.15t}$$
$$u = 5 + 95e^{-0.15(60)}$$
$$u \approx 5.01$$

The temperature of the object will be about $5°C$ after one hour.

60. The population of bacteria present in a certain culture grows according to the law $P = 30e^{0.16t}$ (where the time t is measured in minutes).

a. Find the initial number of bacteria in the culture.
b. Find the number of bacteria in the culture after 1 hour.

Solution

a. Let $t = 0$

$$P = 30e^{0.16(0)}$$
$$P = 30$$

Initially there were 30 bacteria in the culture.

b. Let $t = 60$

$$P = 30e^{0.16(60)}$$
$$P \approx 442,943$$

The number of bacteria in the culture after one hour is about 442,943.

61. Find the intensity of the sound of heavy city traffic that has a loudness level of 90 decibels.

Solution

Given $L = 90$ decibels, $I_0 = 10^{-16}$,

$$L = 10 \log \left(\frac{I}{I_0} \right)$$

$$90 = 10 \log \left(\frac{I}{10^{-16}} \right)$$

$$9 = \log \left(10^{16} I \right)$$

$$10^9 = 10^{16} I$$

- Use definition of logarithm.

$I = 10^{-7}$ watts per square centimeters

The intensity of the sound of heavy traffic is 10^{-7} watts per square centimeters.

PRACTICE TEST

Use a calculator to evaluate each expression accurate to four significant digits.

1. $e^{\sqrt{2}}$

2. $6^{\sqrt{3}}$

Evaluate each of the following logarithms.

3. $\log_5 \frac{1}{25}$

4. $\log_{\frac{9}{4}} \frac{27}{8}$

Write each of the following in terms of logarithms of x, y and z.

5. $\log \dfrac{x^4}{(y-2)^2 z^3}$

6. $\log \sqrt{x}\,(y-1)^3 z^5$

Write each of the following as a single logarithm.

7. $3\log_2 x - 2\log_2(y+2) - 4\log_2(z-5)$

8. $2\ln x - 3\ln y - 5\ln z$

Sketch the graph of each function using graphing techniques. State the domain and range of each function.

9. $f(x) = 2^x + 3$

10. $f(x) = \log_3(x-2)$

Solve each of the following equations.

11. $4^{3x} = \frac{1}{4}$

12. $e^{x+1} = 11$

13. $3^{2x-1} = 25$

14. $\log(3x-1) - \log(4x+1) = 0$

15. $\log_4(x-3) + \log_4(x+3) = 2$

16. $\ln e^{2x+3} = 7$

17. Suppose $\$1,200$ is invested at an interest rate of 7.5% per year. Find the amount after 10 years if the interest is compounded quarterly.

18. How long will it take $1,500 to double if it is invested in an account that earns 8% interest per year compounded continuously?

19. The population of a region grows exponentially according to the function

$$f(t) = 6500\,(1.12)^t \text{ for } 0 \leq t \leq 5.$$

Find the population of the region when t is 4.

20. Write each logarithmic equation as an exponential equation.

a. $\log_3\left(\frac{1}{27}\right) = -3$

b. $\log(0.0001) = -4$

ANSWERS

CHAPTER 1

PRACTICE TEST

1. -16 **2.** $5^{-\frac{1}{4}}$ **3.** 2 **4.** $\frac{x}{y}$ **5.** $\frac{y^{\frac{1}{6}}}{x^{\frac{1}{6}}}$ **6.** $\frac{9y^2}{4x^2}$

7. $-\frac{5y^3}{x^5}$ **8.** $x\sqrt{2}$ **9.** $3x^2y^4\sqrt[3]{3}$ **10.** $\frac{3x\sqrt[3]{4y^2}}{2y}$

11. $\frac{5x^2\sqrt{2y}}{2y}$ **12.** $8x^3y^2$ **13.** $x+2-2\sqrt{x+1}$

14. $2-\sqrt{6}-2i\sqrt{2}-i\sqrt{3}$ **15.** $4+i$ **16.** $48+4i$

17. $-\frac{18}{13}+\frac{12}{13}i$ **18.** i **19.** $\frac{6}{y-1}$ **20.** $\frac{3x+1}{x+1}$

21. $\frac{4}{(y-1)(y+1)}$ **22.** $\frac{y-5}{3y+2}$

CHAPTER 2

PRACTICE TEST

1. $\{3\}$ **2.** $\{-36\}$ **3.** no solution **4.** $a=\frac{bf}{b-f}$

5. $x_1=\frac{y_1-y+mx}{m}$ **6.** two distinct real roots **7.** $\{-1,-4\}$

8. $\left\{-1,\frac{27}{8}\right\}$ **9.** $\{-1\}$ **10.** $11\frac{1}{4}$ hours

11. \$20,000 @ 6% and \$15,000 @ 8% **12.** 7.2 oz.

13. $(-\infty,-5]\cup[3,\infty)$ **14.** $\{-11,7\}$ **15.** $\left[-1,\frac{7}{3}\right]$ **16.** no solution

17. $(-\infty,\infty)$

CHAPTER 3

PRACTICE TEST

1. $\sqrt{2}$; $\left(-\frac{5}{2},-\frac{1}{2}\right)$ **2.** none, $(0,\pm3)$ **3.** center $(3,-5)$, radius 4

CHAPTER 7

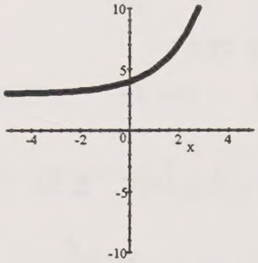
PRACTICE TEST

1. 4.113 **2.** 22.27 **3.** -2 **4.** $\frac{3}{2}$

5. $4\log x - 2\log(y-2) - 3\log z$ **6.** $\frac{1}{2}\log x + 3\log(y-1) + 5\log z$

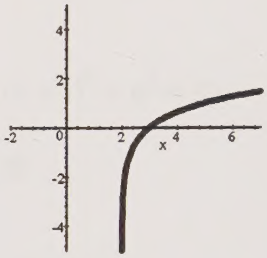
7. $\log_2 \dfrac{x^3}{(y+2)^2(z-5)^4}$ **8.** $\ln \dfrac{x^2}{y^3 z^5}$

9.



Domain: $(-\infty, \infty)$
Range: $(3, \infty)$

10.



Domain: $(2, \infty)$
Range: $(-\infty, \infty)$

11. $-\frac{1}{3}$ **12.** 1.397 **13.** 1.964 **14.** no solution **15.** 5

16. 2 **17.** \$2, 522.82 **18.** 9 years **19.** 10, 228 **20. a.** $\frac{1}{27} = 3^{-3}$

b. $0.0001 = 10^{-4}$

$2(3X-1)$ $4X^2$

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ISBN 0-618-19350-2
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